The discount function

The discount function describes the present value at time $t_0$ of a unit cash flow at time $t$.

$$P(t_0, t)$$

In most cases, $t_0$ is the current date and is sometimes dropped for notational convenience. The remaining variable $t$ then refers to the time between $t_0$ and $t$.

Discount Function when rate = 10%

At $t = 0$, the discount function always has the value 1. The discount function is monotonically decreasing, which corresponds to stipulation that interest rates are always positive. It never reaches zero since all cash flows, no matter how far in the future they are paid, are always worth more than nothing.

The discount function has a mathematical relationship to the spot yield curve, although the "yield curve" is not a well-defined concept. The relationship between the discount function and the annually compounded yield curve, using a day count convention that reflects the actual time between time $t_0$ and $t$ measured in years, can be written as:

$$P(t) = \frac{1}{(1+r_t(t))^t}$$

This formula can be inverted to give

$$r_t(t) = -\frac{\log(P(t))}{t} - 1.$$  

Other frequently used yield and interest rate concepts are listed below, each having a mathematical relationship to the discount function.

**Discount rate, Capitalization rate.**

The rate used to discount a given cash flow in the future to a present value. This rate reflects the time-value of money. Sometimes discount rate = zero coupon rate. The discount rate calculates the present value $X$ at time $t_1$ as:
\[ PV \left( X_n \right) = r_{\text{discount}}^n \cdot X_n \]

The relation to the discount function is given by:
\[ P(t) = 1 - r_{\text{discount}}(t) \cdot t \]

**Simple Rate**

The simple rate is the yield, expressed as percentages of the invested amount. If we receive all interest rate at the end of an investment period we get:
\[ (1 + r_{\text{annual}})' = (1 + r_{\text{simple}} \cdot t) \]

The relation to the discount function is given by:
\[ P(t) = \frac{1}{1 + r_{\text{simple}}(t) \cdot t} \]

**Effective (annual) rate**

The effective rate is the yield expressed as percentages of the invested amount based on a year including the effect of compounding. If we receive interest rate, we have to ask us how often we get payments. If we let \( f \) to be the period, e.g. the number of annual payments we get:
\[ (1 + r_{\text{annual}})' = (1 + \frac{r_f}{f})^{\frac{1}{f}} \]
\[ (1 + r_{\text{annual}})' = (1 + \frac{r_{\text{quarterly}}}{4})^{4t} \]

If we use the par rate of a bond paying the coupon rate \( c_{\text{par}} \) with \( f \) annual coupons in \( n \) years, we get:
\[
100 = \sum_{i=1}^{n f} \left( r_i^{\text{discount}} \cdot \frac{c_{\text{par}}}{f} \right) + \left( r_i^{n f \text{discount}} \cdot 100 \right) \Rightarrow r_{\text{par}} = c_{\text{par}} = f \cdot \frac{100}{\sum_{i=1}^{n f} r_i^{\text{discount}}} \cdot \left( 1 - r_i^{n f \text{discount}} \right)
\]

In continuous compounding this is expressed as:
\[(1 + \frac{r_f}{f})^{f \cdot t} \rightarrow \infty \Rightarrow (1 + r_{\text{annual}})' = e^{r \cdot t}\]

The annual rate is related to the discount function as:
The semi-annual is related to the discount function as:

\[ P(t) = \frac{1}{\left(1 + \frac{r_s(t)}{2}\right)^{2t}} \]

and the N-annual as:

\[ P(t) = \frac{1}{\left(1 + \frac{r_n(t)}{n}\right)^{nt}} \]

The continuous compounding is given as:

\[ P(t) = e^{-c(t)t} \]

Each of these formulae can be inverted in the same way as for the annually compounded interest rate. The formulas also define the implicit relationship between the different interest rate types. Since there is a mathematical relationship between the concepts "discount function" and "yield curve", both of these will be used in this text when we describe the information necessary to perform zero coupon pricing.

**Spot rate**

The spot rate is defined as the theoretical profit given by a zero coupon bond. We use this rate when we calculate the amount we will get at time \( t_1 \) (in the future) if we invest \( X \) today (i.e. at time \( t_0 \)):

\[ X_t = (1 + r_{spot})^{t_1} X_t \]

\[ PV\left(X_t\right) = \frac{1}{(1 + r_{spot})^{t_1}} X_t \]

where \( PV(X_t) \) is the present value of \( X_t \). The relation between the spot rate and the discount function is:

\[ P(t) = \frac{1}{1 + r_{spot}(t)} \]

**Forward rate**

From the yield curve describing the interest rates that apply between the current date and a future date, it is possible to determine an implied forward rate, i.e. the rate that "should" apply between two future dates. The formula for implied forward rates is based on an arbitrage
argument, where the rate for a specific nominal amount between two future dates can be
locked in by borrowing and lending at the current rates to the future dates.

A projection of the future interest rate, from one time to another, calculated from the spot rate
(see above) or a yield curve. The relationship between the spot rate and the forward rate is
given by:

\[
(1 + r_{t_1}^{\text{spot}})^t_1 \cdot (1 + r_{t_2-t_1}^{\text{forward}})^{t_2-t_1} = (1 + r_{t_2}^{\text{spot}})^{t_2} \Rightarrow r_{t_2-t_1}^{\text{forward}} = \frac{(1 + r_{t_2}^{\text{spot}})^{t_2}}{(1 + r_{t_1}^{\text{spot}})^{t_1}}^{\frac{1}{t_2-t_1}} - 1
\]