Interest Rate Theory

1. (a) Consider the Vasiček model

$$dr = (b - ar)dt + \sigma dV,$$

where a, b and σ are constants, we assume that $a > 0$ and V is a Q-Wiener process.

i. Derive an explicit solution for the SDE and determine the distribution of r when $r(0) = r_0 =$ constant.

ii. As $t \to \infty$, the distribution of r tends to a limiting distribution. Which? From the limiting distribution it can be seen that in the limit r will oscillate around its mean reversion level b/a.

iii. Now assume that $r(0)$ is a stochastic variable, which is independent of the Wiener process W, and which has the limiting distribution obtained in (b). Show that this implies that $r(t)$ has the limit distribution from (b), for all t. We have thus found the stationary distribution for the Vasiček model.

iv. Check that the density function of the limiting distribution solves the time invariant Fokker-Planck equation with the $\frac{\partial}{\partial t}$-term equal to zero.

(b) The purpose of the following exercise is to indicate why the CIR model is connected with squares of linear diffusions. Let Y be given as the solution to the following SDE

$$dY = (2aY + \sigma^2)dt + 2\sigma\sqrt{Y}dW.$$

Define the process Z by $Z(t) = \sqrt{Y(t)}$. It turns out that Z satisfies a stochastic differential equation. Which? What is the distribution of Z?

2. Consider an interest model described (under Q) by

$$dr = \mu(t, r)dt + \sigma(t, r)dV.$$

(a) Define what is meant by an *Affine Term Structure* (ATS), and derive conditions which are sufficient to guarantee the existence of an ATS for the model above.

(b) Now consider the Ho-Lee model

$$dr = \Phi(t)dt + \sigma dV.$$

Show explicitly how this model can be fitted to an initially observed price curve $\{p^\star(0, T);\ T \geq 0\}$ which is assumed to be smooth enough.

3. Consider the following simple (and extremely unrealistic) model for the short rate under a fixed martingale measure Q

$$dr(t) = \alpha dt + \sigma dV(t),$$

where α and σ are known deterministic constants, and V is a Q-Wiener process. Determine the arbitrage free price at time zero $\Pi[0; X]$ for the T-contract X

defined by $X = r(T)$. The value of the short rate at time zero is denoted by r_0. You are assumed to know the observed value of the bond price $p(0,T)$, and the answer is allowed to be expressed in terms of this price and other given data above.

4. It is often considered reasonable to demand that a forward rate curve always has a horizontal asymptote, i.e. that $\lim_{T\to\infty} f(t,T)$ exists for all t (the limit will obviously depend upon t and $r(t)$). The object of this exercises is to show that the Ho-Lee model is not consistent with such a demand.

   (a) Compute the explicit formula for the forward rate curve $f(t,T)$ for the Ho-Lee model (fitted to the initial term structure).

   (b) Now assume that the initial term structure indeed has a horizontal asymptote, i.e. that $\lim_{T\to\infty} f^*(0,T)$ exists. Show that this property is not respected by the Ho-Lee model, by fixing an arbitrary time t, and showing that $f(t,T)$ will be asymptotically linear in T.

5. Consider a bond market, with bond prices denoted by $p(t,T)$. Also consider the following forward rate model, under a martingale measure Q.

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t), \\ f(0,T) &= f^*(0,T), \end{aligned}$$

where $f^*(0,T)$ are the observed forward rates at $t = 0$.

   (a) Give the formula which **defines** the forward rates in term of bond prices. There is an economic interpretation of the forward rates. Give this interpretation.

   (b) Explain verbally why it is not possible to specify both the volatility structure $\sigma(t,T)$ and the drift structure $\alpha(t,T)$ freely.

   (c) State and derive the Heath-Jarrow-Morton drift condition, which gives α in terms of σ.

6. A **consol bond** is a bond which forever pays a constant continuous coupon. We normalize the coupon to unity, so over every interval with length dt the consol pays $1 \cdot dt$. No face value is ever paid. The price $C(t)$, at time t, of the consol is the value of this infinite stream of income, and it is obviously (why?) given by

$$C(t) = \int_t^\infty p(t,s)ds$$

Now assume that bond price dynamics under a martingale measure Q are given by

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)v(t,T)dW(t),$$

where W is a vector valued Q-Wiener process. Show (heuristically) that the consol dynamics are of the form

$$dC(t) = [C(t)r(t) - 1]dt + \sigma_C(t)dW(t),$$

where

$$\sigma_C(t) = \int_t^\infty p(t,s)v(t,s)ds.$$

7. Consider a bond market under a fixed martingale measure Q. Let $r(t)$ denote the short rate. We will now look at an interest rate derivative known as the interest rate swap. This is basically a scheme where you exchange a payment stream at a fixed rate of interest, known as the swap rate, for a payment stream at a floating rate. There are many versions of interest rate swaps and we will study some of them below. In all cases we assume that $t = 0$, and that a number of dates $T_1 < T_2 < \ldots < T_M$ have been fixed.

   The swap rate R is assumed to be a constant, which is specified at $t = 0$. Suppose that R is given, then the payment streams for the different versions of the interest rate swap are as follows.

   (a) **Fixed rate - short rate**

   At time $t = 0$ K SEK is invested in the risk free asset, with stochastic short rate, on your account.

   At time $t = T_1$ you receive interest payment on these K SEK, and then the original amount of K SEK is reinvested at the short rate. If for example $K = 100$ and the capital has grown to 110 at time T_1, you receive 10 SEK at $t = T_1$, whereupon 100 SEK are reinvested.

   At time $t = T_2$ you receive interest payment on the at time T_1 invested K SEK, and then the original amount of K SEK is reinvested at the short rate.

   In the same manner K SEK are invested in the risk free asset at every time T_n, and at time T_{n+1} you receive an interest payment of the interest earned over the interval $[T_n, T_{n+1}]$. Specifically, at time T_M you receive an interest payment of the interest earned over the interval $[T_{M-1}, T_M]$ (but not the K SEK).

   These are the payments you receive according to the contract. The payments you have to make are by definition the following.

   For each interval $[T_n, T_{n+1}]$ you invest K SEK at time T_n at the deterministic rate R. At time T_{n+1} you make an interest payment of the interest earned, and you reinvest the K SEK - again at the deterministic rate R.

   Your task is to determine the value of the swap rate R which makes the value at $t = 0$ of the contract described above 0 SEK.

   (b) **Fixed rate - continuously compounded spot rate**

   The payments to be made are exactly as in (a). The only difference in the payments you receive is that the K SEK invested on your account is no longer invested at the short rate, but at the continuously compounded spot rate for the current period. This means that at time T_n the K SEK are invested at the rate $R(T_n, T_{n+1})$. The rate is thus fixed in each interval, but different intervals will typically have different rates.

   Your task is again to determine the value of the swap rate R which makes the value at $t = 0$ of the contract described above 0 SEK.

(c) **Fixed rate - short rate - continuous payments**

This contract is the same as in (a) except we let the time periods be of infinitesimal length. This means that there will be a continuous flow of payments. Your task is once more to determine the value of the swap rate R which makes the value at $t = 0$ of the contract described above 0 SEK. This task can be solved either by letting $T_n = n \cdot h$ and then letting h tend to zero, or by modeling the new situation directly.

Change of Numeraire

1. Suppose that the short rate r has the following dynamics under a martingale measure Q

$$dr = \mu(t,r)dt + \sigma(t,r)dV.$$

(a) Let Q be the martingale measure above and define the forward measure Q^T by

$$\frac{dQ^T}{dQ} = \frac{\exp\left\{-\int_0^T r_s ds\right\}}{E^Q\left[\exp\left\{-\int_0^T r_s ds\right\}\right]}. \tag{1}$$

Let L^T be the likelihood process generated by the change of measure (1) over the time interval $[0,T]$. In other words let

$$L_t^T = \frac{dQ^T}{dQ}, \qquad \text{on } \mathcal{F}_t.$$

Derive an expression for L_t^T in terms of bond prices and the price of the risk free asset.

(b) As is well-known the bond price dynamics under Q is of the form

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dV(t).$$

Determine dL_t^T in terms or the bond price dynamics above.

(c) Now consider the following simple short rate model (under Q)

$$dr_t = \alpha dt + \sigma dV_t,$$

where α and σ are assumed to be known constants. Compute the arbitrage free price at time t of a contract, which at time T pays out r_T^2 SEK.

2. Consider a financial market, where prices are not necessarily driven by Wiener processes. In the model there is a risk free asset B with deterministic interest rate r. We assume that there exists an "ordinary" martingale measure Q, that is, Q is a martingale measure when B is used as numeraire. This means that under Q all processes of the form

$$\frac{\Pi(t)}{B(t)},$$

where $\Pi(t)$ is the price of an (arbitrary) traded asset, are martingales. Now assume that the process Y is the price process of an asset, which from now on is fixed. Now suppose that we want to use Y as numeraire instead of B. Denote by Q^* the martingale measure associated with the numeraire Y and by E^* the expectation under Q^*.

(a) Let X be a contingent T-claim, and denote by $\Pi_t[X]$ the arbitrage free price of X at time t. Determine $\Pi_t[X]$ in terms of a conditional expectation under Q^*.

(b) Fix a point in time T, and determine the Radon-Nikodym derivative

$$L(T) = \frac{dQ^*}{dQ}, \qquad \text{on } \mathcal{F}_T.$$

Remark: This exercises consists of two parts, where one is to present a suggestion for $L(T)$, and the other is to show that the measure induced by the suggested transformation actually is a martingale measure when Y is used as numeraire, i.e that all processes of the form $\Pi(t)/Y(t)$ are Q^*-martingales.

Hint: When looking for $L(T)$ it may be fruitful to use (a) and consider a contract X of the form

$$X = Z \cdot Y(T).$$

(c) Suppose now that Y has a stochastic differential, which is given by

$$dY = \alpha Y dt + \sigma Y dW,$$

under the objective measure P. Determine the Q-dynamics of the Radon-Nikodym derivative for this case, i.e. determine dL under Q.

(d) Let S be the price process of a traded asset with P-dynamics given by

$$dS = \beta S dt + \delta S dW^*,$$

where W^* is a Wiener process independent of W. Consider the contingent claim $X = Y(T)S(T)$. Compute $\Pi_t[X]$.

3. Consider a market on which there exists an equivalent martingale measure Q. Under Q the short rate is assumed to satisfy the following SDE

$$dr_t = a(t)dt + b_1(t)dW_t^1 + b_2(t)dW_t^2,$$

where a, b_1 and b_2 are deterministic functions of time and W^1 and W^2 are two independent Q-Wiener processes.

(a) Show that the price process $p(t,T)$ of a zero-coupon bond maturing at T has the following dynamics under Q

$$dp(t,T) = r_t p(t,T)dt + v_1(t,T)p(t,T)dW_t^1 + v_2(t,T)p(t,T)dW_t^2.$$

Give explicit expressions for v_1 and v_2 in terms of the given functions a, b_1 and b_2.

Hint: This model will give rise to an affine term structure, i.e.

$$p(t,T) = \exp\{A(t,T) - B(t,T)r_t\},$$

where A and B solve the following ordinary differential equations

$$\begin{cases} B_t &= -1 \\ B(T,T) &= 0 \end{cases} \quad \text{and} \quad \begin{cases} A_t &= aB - \frac{1}{2}b_1^2B^2 - \frac{1}{2}b_2^2B^2 \\ A(T,T) &= 0 \end{cases}.$$

If you use this you should show it. One way of doing this is to go through the following steps:

i. Since bond prices are given by the formula

$$p(t,T) = E^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}\middle|\mathcal{F}_t\right],$$

the Markov property yields $p(t,T) = F(t,r_t,T) = F^T(t,r_t)$. Let S_0 denote the risk free asset. Now use Itô's formula on $F^T(t,r_t)/S_0(t)$ and the fact that $F^T(t,r_t)/S_0(t)$ is a Q-martingale to derive the term structure equation for this model (do not forget to give a boundary condition).

ii. Check that the proposed affine term structure

$$F(t,r,T) = \exp\{A(t,T) - B(t,T)r\},$$

satisfies the term structure equation derived in step (i).

(b) Consider a firm whose market value V is described by the following SDE (under Q)

$$dV_t = r_t V_t dt + \sigma_V^1 V_t dW_t^1 + \sigma_V^2 V_t dW_t^2.$$

Here σ_V^1 and σ_V^2 are assumed to be constants.

Suppose that the firm has issued K bonds maturing at T (T is a fixed point in time), each promising to pay 1 SEK at maturity unless the value V_T of the firm is not large enough to cover the debt. Should the value not cover the debt, the firm will go into liquidation and each bond holder receives a default payment of V_T/K SEK. Let $u(t,T)$ denote the price of one of the firms bonds at time t.

Show that the default risk premium $p(0,T) - u(0,T)$ can be written on the following form

$$p(0,T) - u(0,T) = \frac{p(0,T)}{K}\Pi_P(T,M).$$

Here $\Pi_P(T,M)$ denotes the price of a European put option with time of maturity T and strike price M. Part of the exercise is to specify the "underlying" object on which the option is made out, what the strike price M is, and which interest rate and (time-dependent) volatilities should be used when calculating $\Pi_P(T,M)$.

Hint: Forward measures may be of use.

4. Let $c(t,T,K,S)$ denote the price at time t of a European call option with time of maturity T and strike price K, on an underlying S-bond ($T < S$).

Consider the Ho-Lee model

$$dr = \phi(t)dt + \sigma dV,$$

where ϕ is a deterministic function of time, σ is a constant and V is a Q-Wiener process. Show that

$$c(t,T,K,S) = p(t,S)N(d) - p(t,T)\cdot K \cdot N(d - \sigma_p),$$