Solution to exam in Analytical Finance II, January 2008.

1. See the Lecture Notes.

2. We recall

$$f(t,T) = -\frac{\partial}{\partial T}\ln p(t,T)$$
$$p(t,T) = e^{-\int_t^T f(t,s)ds}$$

From these we immediately have the answers to (a) and (b):

$$(a) \quad y(t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds$$
$$(b) \quad f(t,T) = \frac{\partial}{\partial T} y(t,T)\left(T-t\right) + y(t,T)$$

The idea, exactly as for HJM theory, is to use the fact that, under Q, the local rate of return of all bonds equals the short rate. We recall yield dynamics under Q as

$$dy(t,T) = a(t,T)dt + b(t,T)dW(t)$$

We can write the bond prices as

$$p(t,T) = e^{Z(t,T)}$$

with $Z(t,\ T) = -y(t,\ T)\cdot(T-t)$ From Ito we have

$$\begin{aligned} dZ(t,T) &= -(T-t)dy(t,T) + y(t,T)dt \\ &= \left\{y(t,T) - a(t,T)(T-t)\right\}dt - b(t,T)(T-t)dW(t) \end{aligned}$$

Again from Ito we have

$$dp(t,T) = p(t,T)dZ + \frac{1}{2}p(t,T)\left(dZ\right)^2$$

Inserting the Z-dynamics we obtain

$$\begin{aligned} dp(t,T) = p(t,T)&\left\{y(t,T) - a(t,T)(T-t) + \frac{1}{2}b(t,T)^2(T-t)^2\right\}dt \\ &- p(t,T)b(t,T)(T-t)dW(t) \end{aligned}$$

Under Q, the local rate of return of all bonds equals the short rate so, denoting the short rate by r_t and recalling that $r_t = y(t,\ t)$ we have

$$r(t) = y(t,T) - a(t,T)(T-t) + \frac{1}{2}b(t,T)^2(T-t)^2$$

i.e. the drift condition is

$$a(t,T) = \frac{1}{T-t}\left\{ y(t,T) - y(t,t) + \frac{1}{2}b(t,T)^2(T-t)^2 \right\}$$

3. The six month interest rate is given by the bill:

$$98.068 = \frac{100}{\left(1+r_{6m}\right)^{180/360}} \quad \Rightarrow \quad r_{6m} = \left(\frac{100}{98.068}\right)^2 - 1 = 4.000\%$$

With extrapolation we have $r_{2m} = r_{3m} = r_{6m} = 4.000\%$

To get the interest rate at one year and three months, we use one of the bonds:

$$101.436 = \frac{5}{\left(1+r_{3m}\right)^{90/360}} + \frac{105}{\left(1+r_{1y3m}\right)^{1+90/360}} \quad \Rightarrow \quad r_{1y3m} = 7.000\%$$

With use of linear interpolation we can calculate the rate at one year and two months:

$$r_{1y2m} = 4.000\% + \frac{7.000\% - 4.000\%}{1 + \frac{90}{360} - \frac{180}{360}}\left(1 + \frac{60}{360} - \frac{180}{360}\right) = 6.667\%$$

The rate at 2 year and 2 month is given by:

$$102.000 = \frac{6}{\left(1+r_{2m}\right)^{60/360}} + \frac{6}{\left(1+r_{1y2m}\right)^{1+60/360}} + \frac{106}{\left(1+r_{2y2m}\right)^{2+90/360}} \quad \Rightarrow \quad r_{1y2m} = 7.426\%$$

Finally, we get the two year interest rate by interpolation:

$$r_{2y} = 7.000\% + \frac{7.426\% - 7.000\%}{\left(2 + \frac{60}{360}\right) - \left(1 + \frac{90}{360}\right)}\left(2 - \left(1 + \frac{90}{360}\right)\right) = 7.349\%$$

4.

(a) Set $u_t = E[r_t]$. We have that

$$r_t = r_0 + \int_0^t (\alpha - \beta r_s)\, ds + \int_0^t \sigma\sqrt{r_s}\, dB_s.$$

Since the second term is a stochastic integral, with a square integrable integrand, it has mean 0, so

$$u_t = r_0 + E[\int_0^t (\alpha - \beta r_s)\, ds] = \int_0^t E[\alpha - \beta r_s]\, ds = \int_0^t [\alpha - \beta u_s]\, ds.$$

Differentiating, we get an ODE for u_t, which is easily solved, namely

$$du_t = (\alpha - \beta u_t)\, dt,$$

or $u_t' = \alpha - \beta u_t$.
[You didn't have to solve, it but here is how the argument would go: Rewrite it in the form

$$\frac{du_t}{\alpha - \beta u_t} = dt$$

and integrate both sides to get

$$\frac{-1}{\beta} \log\left(\frac{\alpha - \beta u_t}{\alpha - \beta r_0}\right) = t.$$

Solving for u_t yields

$$u_t = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + r_0 e^{-\beta t} \to \frac{\alpha}{\beta}$$

as $t \to \infty$.]

(b) To find the second moment of r_t we do exactly the same thing. The first step is to find the SDE for r_t^2. From the SDE for r_t we have that $d\langle r\rangle_t = \sigma^2 r_t\, dt$. Thus by Itô's lemma,

$$\begin{aligned} dr_t^2 &= 2r_t\, dr_t + d\langle r\rangle_t \\ &= 2r_t(\alpha - \beta r_t)\, dt + 2\sigma r_t^{3/2}\, dB_t + \sigma^2 r_t\, dt. \end{aligned}$$

In other words,

$$r_t^2 = r_0^2 + \int_0^t (2\alpha + \sigma^2) r_s\, ds - 2\beta \int_0^t r_s^2\, ds + 2\sigma \int_0^t r_s^{3/2}\, dB_s.$$

Write $v_t = E[r_t^2]$, so that $v_0 = r_0^2$. Then taking expectations,

$$E[r_t^2] = r_0^2 + \int_0^t (2\alpha + \sigma^2) E[r_s]\, ds - 2\beta \int_0^t E[r_s^2]\, ds.$$

In other words,

$$v_t = r_0^2 + \int_0^t (2\alpha + \sigma^2) u_s\, ds - 2\beta \int_0^t v_s\, ds.$$

Differentiating, we get the ODE for v_t:

$$v_t' = (2\alpha + \sigma^2)u_t - 2\beta v_t.$$

[Again, this could be solved, as follows: The standard method is to use what is called an integrating factor. In other words, to note that

$$\begin{aligned}(v_t e^{2\beta t})' &= e^{2\beta t}(v_t' + 2\beta v_t) \\ &= e^{2\beta t}(2\alpha + \sigma^2)u_t \\ &= (2\alpha + \sigma^2)\Big[e^{\beta t}\Big(r_0 - \frac{\alpha}{\beta}\Big) + e^{2\beta t}\frac{\alpha}{\beta}\Big].\end{aligned}$$

Integrating gives that

$$\begin{aligned}v_t e^{2\beta t} &= v_0 e^{2\beta\cdot 0} + (2\alpha + \sigma^2)\int_0^t \Big[e^{\beta s}\Big(r_0 - \frac{\alpha}{\beta}\Big) + e^{2\beta s}\frac{\alpha}{\beta}\Big]\,ds \\ &= r_0^2 + (2\alpha + \sigma^2)\left[\Big(r_0 - \frac{\alpha}{\beta}\Big)\frac{e^{\beta t} - 1}{\beta} + \frac{\alpha}{\beta}\frac{e^{2\beta t} - 1}{2\beta}\right] \\ &= r_0^2 + (2\alpha + \sigma^2)\left[\Big(\frac{r_0}{\beta} - \frac{\alpha}{\beta^2}\Big)e^{\beta t} + \frac{\alpha}{2\beta^2}e^{2\beta t} + \Big(\frac{\alpha}{2\beta^2} - \frac{r_0}{\beta}\Big)\right].\end{aligned}$$

Thus

$$\begin{aligned}v_t = r_0^2 e^{-2\beta t} &+ (2\alpha + \sigma^2)\left[\Big(\frac{r_0}{\beta} - \frac{\alpha}{\beta^2}\Big)e^{-\beta t} + \frac{\alpha}{2\beta^2} + \Big(\frac{\alpha}{2\beta^2} - \frac{r_0}{\beta}\Big)e^{-2\beta t}\right] \\ &\to (2\alpha + \sigma^2)\frac{\alpha}{2\beta^2}.\end{aligned}$$

The point of the question is that pricing involves finding the state-price density. This may not always be easy, but at least we can extract information about the moments, using the above arguments. It turns out that CIR models produce equilibrium distributions that are gamma distributed, so in fact, the moments characterize the distributions.]

5.) See Lecture Notes

6.) See Lecture Notes