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Examination in MT1470/MMA708 Analytical Finance II
Wednesday 9 of January 2008, 14:30 – 18:30

Examiner: Jan Röman, phone 0708-606 306
You may use: Calculator, pencil, ruler and rubber gum.
General direction: The solution should be well motivated and readable. All notations must be explained.

Remark: Write your national registration number (personnummer) and the number of pages on the first page. Write only one solution on per sheet. Use page numbers and write your name on all pages.

Remark2: There are 6 problems. You don't have to make both of 5 and 6. But if you are close to a limit I will consider both of them.

Good Luck!!

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1. (a) Explain the concept of zero-coupon pricing.

   (b) What is the repo-rate S/N?

   (c) Derive by arbitrage argument the forward rates as function of the spot rates. Assume that the spot rates at times t_1, t_2, … t_n are known.

   (d) Explain dirty price and clean price. Draw a graph of the dirty price where the market rate is lower then the coupon rate.

   (e) Define discount margin.

   (f) What is the minimum numbers of legs in a swap contract? How can a swap have more legs?

   (g) Explain the building blocks in a CPPI.

   (h) What is the Nelson Siegel model?

   (i) What is the relation between the forward prices and the zero coupon bond prices?

   (j) Why/when do you need the Ito calculus in finance?

(k) Why/when do you need Girsanov transformations in finance?

(l) What does the Meta theorem tell us about the market of interest rates?

(m) Define what we mean with CTD.

(n) Which interest rate model can be solved by a trinomial tree?

(o) Who decide the market price of risk?

(p) When do you use the Feynmann-Kac representation in finance?

(q) What do we mean with convexity adjustment?

(r) If the forward rates $f(t, T)$, how do you get the short rate $r(t)$?

(s) Define in words and formula the concept of changing numeraire.

(t) Define in words and formula the forward measure and why the forward measure simplifies the calculations dramatically
……………………………………………………………….………………(10p)

2. Consider a bond market with bond prices $p(t, T)$, and let $f(t, T)$ as usual denote the forward rate at t with maturity T. Define, for each maturity T, the **yield** process $y(t, T)$ by the relation

$$p(t,T) = e^{-y(t,T)\cdot(T-t)}$$

(a) Give a formula for how the yield process $y(t, T)$ is determined in terms of the forward rate processes $\{f(t, s): s \geq t\}$-..……..…..…………………………(2p)
(b) Give a formula for how the forward rate process $f(t, T)$ is determined in terms of yield processes $\{y(t, s): s \geq t\}$.…………………………………………(2p)
(c) Suppose that we specify the following yield dynamics under a martingale measure Q

$$dy(t,T) = a(t,T)dt + b(t,T)dW(t)$$

Here, a and b are adapted processes (for each fixed T). Derive the relevant drift condition, i.e. the relation that must hold between a and b. The standard HJM drift condition for forward rates may be used without proof.
……………………………………………………………….………………(10p)

3. You are going to buy a derivative security with maturity two years from now. Therefore you need to know the two year risk-free interest rate. On the market you find the following government securities used as benchmark:

1.) A government bill with maturity six months traded at the price 98.068.

2.) A government bond with maturity one year and three months with a coupon rate of 5% traded at the price 101.436.

3.) A government bond with maturity two year and two months with a coupon rate of 6% traded at the price 102.000.

All the above securities are quoted in percentage of the face value with the day-count convention 30/360. The bonds are paying one coupon per year.
Your task is to calculate the two year interest rate from the above instruments.
Hint: Bootstrap, interpolation and extrapolation.
……………………………………………………………………………….(10p)

4. This problem concerns the Cox-Ingersoll-Ross interest rate model.

(a) Let

$$dr_t = (\alpha - \beta \cdot r_t)dt + \sigma\sqrt{r_t}dV_t$$

where α, β, σ, and the initial value r_0 are all positive constants.

(a) Write this SDE in integral form, take expectations, and then differentiate with respect to t. Express the result as an ordinary differential equation for $u(t) = \mathrm{E}[r_t]$. In other words, an equation for u'_t in terms of u_t.

(b) Use Ito's lemma to obtain an SDE for r_t^2. Carry out the same procedure as above to obtain an ODE for $v_t = \mathrm{E}[r_t^2]$. In other words, find an equation for v'_t in terms of v_t and u_t.

[Note: these ODEs can actually be solved, though I'm not asking you to do this. This allows one to calculate the mean and variance of r_t, which is useful, as the above is one of the standard models for spot interest rates.]
………………………………………………………………………………..(10p)

5. When we considered the stochastic processes for the short rate, the forward rate and the bond prices we started with the following processes:

$$dr(t) = \mu(t)dt + \sigma(t)dW(t),$$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \text{ and}$$

$$dp(t,T) = m(t,T)p(t,T)dt + v(t,T)p(t,T)dW(t)$$

Here, μ and σ are adapted processes, defined for all times $t \geq 0$. For each fixed T, $m(t, T)$, $v(t, T)$, $\sigma(t, T)$ and $\alpha(t, T)$ are adapted processes for $0 \leq t \leq T$. We also supposed that all the processes above are continuous in t and two times differentiable. Further, we supposed that $v(T, T) = 0$ for all T. This seems to be OK since $p(T, T) = 1$ by definition.

We then have three choices. We can start with

- the dynamics of the short rate dr,
- the dynamics of the forward rates df^T or by
- the dynamics of the bond prices dp^T

Derive the relation between the forward rates (dp^T) and the short rates (df^T). ………………………………………………………………………..……………….(10p)

6. Derive the Equation of the Term Structure of interest rates. ………………………………………………………………………..……………….(10p)