LIBOR versus OIS discounting when bootstrapping the STIBOR swap curve

Introduction

In this document I will try to explain the problems of bootstrapping interbank curves. The interbank curve, in SEK, the STIBOR swap curve is used to calculate the floating payments for swaps in Swedish krona. A similar approach as below can be used in other markets as well. But here we will study the Swedish market.

Before the financial crises, the market used the STIBOR curve also for discounting. Then, the bootstrap process was much simpler than today when the swaps are under collateralization and cleared on clearing houses such as Nasdaq. Now days the STIBOR-curve discounting is replaced with OIS-discounting using over-night index swaps. In Sweden we have no "real" OIS curve, but we have a STINA curve (STIBOR tomorrow next index average) based on STINA swaps.

First, we study how to bootstrap the STIBOR curve under STIBOR discounting. Then we will investigate how to use OIS-discounting using the STINA curve.

We start with the quotes given on the market. These quotes are the fixed STIBOR swap rates at times T_i, where $i = 1, 2, \ldots$. By using these quotes we can calculate (bootstrap) the floating rates (the zero-coupon swap curve) using STIBOR discounting.

As we will see, there is a one-to-one relationship between the fixed rates and the floating rates. The reason is that the sum of the floating rate and the fixed rates (the quotes) are related to each other. If we change the floating rate we have to calculate new fixed rates (quotes) to match the curve. Or, if we change the quotes we need to calculate a new floating rate curve, i.e., making a new bootstrap.

So, in this bootstrap process we ignore how the market quotes are calculated or estimated. This means that we can ignore if they are collateralized under EUR, SEK or not collateralized at all. We just want to know what zero-curve they imply. The quotes are set by the market-makers as prices they agree to be used when buying or selling the specific swaps on the interbank market. Now days, these quotes are said to be quoted under collateralization. This means that the quotes are given under OIS discounting! Some traders say that they quote them under EUR OIS discounting (under EONIA) since most collateral agreements are in EUR. Traders also say that the difference will be just 1-2 basis points.

Bootstrapping STIBOR using STIBOR discounting

To set-up a bootstrap we first need a formula for the forward rates, since they will give the floating rates in swap contracts. The easiest way to describe them is via the discount factors which are the base of all interest rates, annually, simple, money-market or continuous compounded rates.

Deposits

First we deal with the short end of the curve consisting of deposits. We start with the over-night rate (if there any over-night quotes) then the tomorrow-next rate and the deposits, 1 week, 1, 2 and 3 months. The discount factors are given by

$$D_{O/N} = \frac{1}{1 + C_{O/N} \cdot \frac{d_{O/N}}{360}}$$

$$D_{T/N} = \frac{D_{O/N}}{1 + C_{T/N} \cdot \frac{d_{T/N}}{360}}$$

$$D_i = \frac{D_{T/N}}{1 + C_i \cdot \frac{d_i}{360}} \qquad i = \text{1W, 1M, 2M, 3M}$$

were the C's are the given quotes and the d's, the number of days. The continuous zero-rates are calculated as

$$r_i = -100 \cdot \frac{\ln\left(D_i\right)}{\frac{d_i}{365}}$$

The zero rates are commonly given as continuous compounding, Act/365. If the day today is a Friday, $d_{O/N}$ = 3 instead of 1. Same apply for tomorrow next since we use day counting with the Swedish calendar to use the correct business days.

FRAs

Next, we use the Forward Rate Agreements. Since these are forward contracts, their quoted rates are forward rates. Therefore, we need a so-called stub rate. This stub rate shall have Maturity the same date as the Start date of the first FRA contract. We find that rate by (linear) interpolation. Here there are different choices. You can interpolate using the zero-rates, the discount factors or, as Simcorp Dimension the quotes. The zero rate formula is

$$r_{stub}(t) = r(t_0) + \frac{r(T) - r(t_0)}{T - t_0}\left(t - t_0\right)$$

Then, for the FRA we have

$$D_{Stub} = \exp\left\{-r_{Stub} \cdot \frac{T_{Stub}}{365}\right\}$$

$$D_{FRA}^{i} = \frac{D_{FRA}^{i-1}}{1 + C_{FRA}^{i} \cdot \frac{d_{FRA}^{i}}{360}}$$

$$r_{FRA}^{i}(T) = -100 \cdot \frac{\ln\left(D_{FRA}^{i}\right)}{\frac{d_{FRA}^{i}}{365}}$$

The quote is given by

$$C_{FRA}^{i} = \left(\frac{D_{FRA}^{i-1}}{D_{FRA}^{i}} - 1\right) \cdot \frac{360}{d_{FRA}^{i}}$$

Swaps

By an arbitrage argument the following must hold; If we want to calculate the present value of a cash-flow at time T we multiply the cash-flow amount with a discount factor $p(T)$. But, we can also discount the same amount in two steps, first by multiplying with the forward discount factor $p(t, T)$ and then by $p(t)$. Here $0 < t < T$ and $p(t, T)$ is the discount factor between t and T. Therefor we must have

$$p(T) = p(t,T) \cdot p(t)$$

Here, $p(t, T)$ is a forward discount factor between t and T given by the forward rate $f(t, T)$. With simple (money-market) compounding the forward discount factor is given by

$$p(t,T) = \frac{1}{1 + (T - t) \cdot f(t,T)} \tag{1}$$

This is what we normally use for periods shorter than a year. With annual compounding this would be

$$p(t,T) = \frac{1}{\left\{1 + f(t,T)\right\}^{T-t}}$$

Since all cash-flows have a tenor of a year or less we use Eq. (1). We therefore get:

$$p(T) = p(t,T) \cdot p(t) \equiv \frac{p(t)}{1 + (T-t) \cdot f(t,T)} \equiv \frac{p(t)}{1 + \Delta \cdot f(t,T)}$$

Where we define $\Delta = T - t$. We can now express the forward rate above as

$$f(t,T) = \frac{1}{\Delta}\left(\frac{p(t)}{p(T)} - 1\right) = \frac{1}{\Delta}\frac{p(t) - p(T)}{p(T)}$$

The floating payment at time T_i given by the forward rate between T_{i-1} and T_i is given by ($\Delta_i = T_i - T_{i-1}$):

$$\Delta_i \cdot f_i \cdot p(T_i) = \Delta_i \frac{1}{\Delta_i} \frac{p(T_{i-1}) - p(T_i)}{p(T_i)} \cdot p(T_i) = p(T_{i-1}) - p(T_i)$$

where f_i is the forward rate $f(T_{i-1}, T_i)$ between T_{i-1} and T_i. All discount factors above are defined until today. Now, the value of the floating leg is

$$\sum_{i=1}^{n} \Delta_i \cdot f_i \cdot p(T_i) = \sum_{i=1}^{n} \left\{ p(T_{i-1}) - p(T_i) \right\} = p(T_0) - p(T_n) \equiv 1 - p(T)$$

Note that all discount factors are cancelling out each other because of the minus sign except the first one and the last one. Also $p(0) = 1$. The fixed leg is given by (using the same discount factors):

$$\sum_{i=1}^{m} C \cdot \Delta_i \cdot p(T_i) = C \cdot \sum_{i=1}^{m} \Delta_i \cdot p(T_i)$$

where C is the quoted fixed rate. Here the tenor might be different, like 3 months on floating leg and 1 year for the fixed leg. When we enter a swap at the first time, both legs shall have the same value (i.e. the sum of the values of the two legs must be zero) we then get:

$$C = \frac{1 - p(T)}{\sum_{i=1}^{m} \Delta_i \cdot p(T_i)} = \frac{1 - p(T)}{\sum_{i=1}^{m-1} \Delta_i \cdot p(T_i) + \Delta_m \cdot p(T)} \qquad (2)$$

In a bootstrap we use this formula with known quotes C's to recursively calculate the discount factors $p(T)$ for different maturities using:

$$p(T_m) = \frac{1 - C_m \cdot \sum_{i=1}^{m-1} \Delta_i \cdot p(T_i)}{1 + \Delta_i \cdot C_m} \qquad (3)$$

In the above calculations we have ignored that quotes are given with two settlement days and that the short and mid parts of the curves are built from deposits and Forward Rate Agreements (IMM FRAs).

We can now also calculate the continuous compounded interest rates, the zero rates:

$$r(T) = -100 \cdot \frac{\ln\left(p(T)\right)}{T}$$

From Eq. (2) and (3) we see the one-to-one relationship between the discount functions that gives the floating zero-rate and the fixed rate. This can be compared to the one-to-one relationship between the price and yield-to-maturity (YTM) for bonds. In Sweden they are quoted in YTM and in most other countries in (clean) price. Also compare the one-to-one relationship with option prices and implied volatility. As soon you know the volatility you know the price and the risk neutral probability measure.

Bootstrapping STIBOR using OIS (STINA) discounting

In a risk-free world we would value the swaps using OIS discounting. We then see a problem with the calculations above; if we value a swap with the curve above, just after the deal, using OIS discounting the value is not zero! This is, if we take the floating cash-flows using the STIBOR curve and the fixed cash-flows from the quotes and discount the sum of the legs, the total value will not be zero.

Therefor we cannot bootstrap with the formulas above. In OIS discounting, the floating leg is calculated as

$$\sum_{i=1}^{n} \Delta_i f_i^{STIBOR} p^{OIS}(T_i)$$

where the cash-flow are given by the forward rate using the STIBOR curve, but we need to discount with the OIS curve. Similarly, the fixed leg must be discounted using the OIS curve:

$$C \cdot \sum_{i=1}^{n} \Delta_i p^{OIS}(T_i)$$

If the sum of these legs is zero, we must have

$$C \cdot \sum_{i=1}^{n} \Delta_i p^{OIS}(T_i) - \sum_{i=1}^{n} \Delta_i f_i^{STIBOR} p^{OIS}(T_i) = 0$$

For simplicity, we now write p^{STIBOR} as p. We now get the forward STIBOR rate $f^{STIBOR}(t, T)$ by

$$f^{STIBOR}(t,T) = \frac{1}{\Delta} \frac{p(t) - p(T)}{p(T)}$$

We then have

$$C \cdot \sum_{i=1}^{n} \Delta_i p^{OIS}(T_i) - \sum_{i=1}^{n} \frac{p(T_{i-1}) - p(T_i)}{p(T_i)} p^{OIS}(T_i) = 0$$

The last sum can be rewritten as

$$\begin{aligned}
\sum_{i=1}^{n} \left\{ \frac{p(T_{i-1})}{p(T_i)} - 1 \right\} p^{OIS}(T_i) &= \sum_{i=1}^{n} \left\{ \frac{p(T_{i-1})}{p(T_i)} p^{OIS}(T_i) - p^{OIS}(T_i) \right\} \\
&= \sum_{i=1}^{n} \frac{p(T_{i-1})}{p(T_i)} p^{OIS}(T_i) - \sum_{i=1}^{n} p^{OIS}(T_i) \\
&= \sum_{i=1}^{n-1} \frac{p(T_{i-1})}{p(T_i)} p^{OIS}(T_i) - \sum_{i=1}^{n} p^{OIS}(T_i) + \frac{p(T_{n-1})}{p(T_n)} p^{OIS}(T_n)
\end{aligned}$$

We now get

$$C \cdot \sum_{i=1}^{n} \Delta_i p^{OIS}(T_i) - \sum_{i=1}^{n-1} \left\{ \left(\frac{p(T_{i-1})}{p(T_i)} - 1 \right) p^{OIS}(T_i) \right\} - \frac{p(T_{n-1})}{p(T_n)} p^{OIS}(T_n) + p^{OIS}(T_n) = 0$$

So

$$\frac{p(T_{n-1})}{p(T_n)} p^{OIS}(T_n) = p^{OIS}(T_n) + C_n \cdot \sum_{i=1}^{n} \Delta_i p^{OIS}(T_i) - \sum_{i=1}^{n-1} \left\{ \left(\frac{p(T_{i-1})}{p(T_i)} - 1 \right) p^{OIS}(T_i) \right\}$$

Finally we get

$$p(T_n) = \frac{p^{OIS}(T_n) \cdot p(T_{n-1})}{p^{OIS}(T_n) + C_n \cdot \sum_{i=1}^{n} \Delta_i p^{OIS}(T_i) - \sum_{i=1}^{n-1} \left\{ \left(\frac{p(T_{i-1})}{p(T_i)} - 1 \right) p^{OIS}(T_i) \right\}}$$

This is the formula we need to use to calculate the STIBOR discount factors under OIS discounting. To get the discount factors for the OIS curve we first have to bootstrap the OIS (STINA) curve from the STINA quotes.

The repricing (calculating of the quotes) using OIS discounting is given by:

$$C = \frac{\sum_{i=1}^{n} \left\{ \left(\frac{p(T_{i-1})}{p(T_i)} - 1 \right) p^{OIS}(T_i) \right\}}{\sum_{i=1}^{n} \Delta_i p^{OIS}(T_i)} = \frac{\sum_{i=1}^{n} \left\{ \left(p(T_{i-1}) - p(T_i) \right) \frac{p^{OIS}(T_i)}{p(T_i)} \right\}}{\sum_{i=1}^{n} \Delta_i p^{OIS}(T_i)}$$

The FRA can be treated as

$$P = N \cdot \tau \cdot \left[L_\tau(t, T_1, T_2) - X \right] \cdot p_D(t, T_2)$$

$$L_\tau(t, T_1, T_2) = \frac{1}{\tau} \left(\frac{p_\tau(t, T_1)}{p_\tau(t, T_2)} - 1 \right)$$

so

$$P = N \cdot \left[\frac{p_\tau(t, T_1)}{p_\tau(t, T_2)} - 1 - \tau \cdot X \right] \cdot p_D(t, T_2)$$

In single curve framework, the last formulae would be expressed as

$$P = N \cdot \left[p_\tau(t, T_1) - p_\tau(t, T_2) \right] \cdot \left(1 + \tau \cdot X \right)$$

But to use the formula above, we need to know the strike. So we have no possibility to compensate the FRA rate for OIS discounting. With known discount factors, the FRA quotes are given by

$$C_{FRA}^{i}=\left(\frac{D_{FRA}^{i-1}}{D_{FRA}^{i}}-1\right)\cdot\frac{360}{d_{FRA}^{i}}$$

independent of the OIS rate.

Bootstrapping the OIS STINA curve

When bootstrapping the STINA curve we have typically quotes given by {T/N, 1W, 1M, 2M, 3M, 6M, 9M, 1Y, 15M, 18M, 2Y, 3Y, 4Y, 5Y, 19Y, 15Y, 20Y, 30Y}. Remark long maturities are not liquid.

For rates up to a year, we use

$$D_{T/N}=\frac{1}{1+C_{T/N}\cdot\frac{d}{360}}$$

and

$$D_{T}=\frac{D_{T/N}}{1+C_{T}\cdot\frac{d_{T}-d_{T/N}}{360}}$$

for the discount factors and

$$r_{T}=100\left\{\left(\frac{1}{D}\right)^{\frac{365}{T}}-1\right\}$$

for the zero-rates. These formulas agree with the result from Nasdaq using their quotes and comparing to their discount curve, discount factors and zero rates. For maturities above a year, the following formula are used for the discount factors depending on the length of the tenor

$$D_{T}=\frac{D_{T/N}-C_{T}\sum_{t}D_{t}\left(\frac{d_{t}-d_{t-1}}{360}\right)}{1+\frac{d_{T}-d_{T-1}}{360}C_{T}}$$

for one year tenors and

$$D_{T}=\frac{D_{T/N}-C_{T}\sum_{t}D_{t}\left(\frac{d_{t}-d_{t-1}}{360}\right)}{\left(1+C_{T}\right)^{\frac{d_{T}-d_{T-1}}{360}}}$$

for tenors longer than a year. The zero rates are given by the same formula as above.