Chapter 28

Interest Rate Derivatives: Models of the Short Rate

SOLUTIONS TO QUESTIONS AND PROBLEMS

Problem 28.1.

Equilibrium models usually start with assumptions about economic variables and derive the behavior of interest rates. The initial term structure is an output from the model. In a no-arbitrage model the initial term structure is an input. The behavior of interest rates in a no-arbitrage model is designed to be consistent with the initial term structure.

Problem 28.2.

In Vasicek's model the standard deviation stays at 1%. In the Rendleman and Bartter model the standard deviation is proportional to the level of the short rate. When the short rate increases from 4% to 8% the standard deviation increases from 1% to 2%. In the Cox, Ingersoll, and Ross model the standard deviation of the short rate is proportional to the square root of the short rate. When the short rate increases from 4% to 8% the standard deviation of the short rate increases from 1% to 1.414%.

Problem 28.3.

If the price of a traded security followed a mean-reverting or path-dependent process there would be a market inefficiency. The short-term interest rate is not the price of a traded security. In other words we cannot trade something whose price is always the short-term interest rate. There is therefore no market inefficiency when the short-term interest rate follows a mean-reverting or path-dependent process. We can trade bonds and other instruments whose prices do depend on the short rate. The prices of these instruments do not follow mean-reverting or path-dependent processes.

Problem 28.4.

In a one-factor model there is one source of uncertainty driving all rates. This usually means that in any short period of time all rates move in the same direction (but not necessarily by the same amount). In a two-factor model, there are two sources of uncertainty driving all rates. The first source of uncertainty usually gives rise to a roughly parallel shift in rates. The second gives rise to a twist where long and short rates moves in opposite directions.

Problem 28.5.

No. The approach in Section 28.4 relies on the argument that, at any given time, all bond prices are moving in the same direction. This is not true when there is more than one factor.

Problem 28.6.

In Vasicek's model, $a = 0.1$, $b = 0.1$, and $\sigma = 0.02$ so that

$$B(t, t+10) = \frac{1}{0.1}(1 - e^{-0.1\times 10}) = 6.32121$$

$$A(t, t+10) = \exp\left[\frac{(6.32121 - 10)(0.1^2 \times 0.1 - 0.0002)}{0.01} - \frac{0.0004 \times 6.32121^2}{0.4}\right]$$
$$= 0.71587$$

The bond price is therefore $0.71587e^{-6.32121\times 0.1} = 0.38046$.

In the Cox, Ingersoll, and Ross model, $a = 0.1$, $b = 0.1$ and $\sigma = 0.02/\sqrt{0.1} = 0.0632$. Also

$$\gamma = \sqrt{a^2 + 2\sigma^2} = 0.13416$$

Define

$$\beta = (\gamma + a)(e^{10\gamma} - 1) + 2\gamma = 0.92992$$

$$B(t, t+10) = \frac{2(e^{10\gamma} - 1)}{\beta} = 6.07650$$

$$A(t, t+10) = \left(\frac{2\gamma e^{5(a+\gamma)}}{\beta}\right)^{2ab/\sigma^2} = 0.69746$$

The bond price is therefore $0.69746e^{-6.07650\times 0.1} = 0.37986$.

Problem 28.7.

Using the notation in the text, $s = 3$, $T = 1$, $L = 100$, $K = 87$, and

$$\sigma_P = \frac{0.015}{0.1}(1 - e^{-2\times 0.1})\sqrt{\frac{1 - e^{-2\times 0.1\times 1}}{2\times 0.1}} = 0.025886$$

From equation (28.6) $P(0,1) = 0.94988$, $P(0,3) = 0.85092$, and $h = 1.14277$ so that equation (28.20) gives the call price as call price is

$$100 \times 0.85092 \times N(1.14277) - 87 \times 0.94988 \times N(1.11688) = 2.59$$

or \$2.59.

Problem 28.8.

As mentioned in the text, equation (28.20) for a call option is essentially the same as Black's model.[2] By analogy with Black's formulas corresponding expression for a put option is

$$KP(0,T)N(-h+\sigma_P) - LP(0,s)N(-h)$$

In this case the put price is

$$87 \times 0.94988 \times N(-1.11688) - 100 \times 0.85092 \times N(-1.14277) = 0.14$$

Since the underlying bond pays no coupon, put–call parity states that the put price plus the bond price should equal the call price plus the present value of the strike price. The bond price is 85.09 and the present value of the strike price is $87 \times 0.94988 = 82.64$. Put–call parity is therefore satisfied:

$$82.64 + 2.59 = 85.09 + 0.14$$

Problem 28.9.

As explained in Section 28.4, the first stage is to calculate the value of r at time 2.1 years which is such that the value of the bond at that time is 99. Denoting this value of r by r^*, we must solve

$$2.5A(2.1,2.5)e^{-B(2.1,2.5)r^*} + 102.5A(2.1,3.0)e^{-B(2.1,3.0)r^*} = 99$$

where the A and B functions are given by equations (28.7) and (28.8). The solution to this is $r^* = 0.066$. Since

$$2.5A(2.1,2.5)e^{-B(2.1,2.5)\times 0.066} = 2.43473$$

and

$$102.5A(2.1,3.0)e^{-B(2.1,3.0)\times 0.063} = 96.56438$$

the call option on the coupon-bearing bond can be decomposed into a call option with a strike price of 2.43473 on a bond that pays off 2.5 at time 2.5 years and a call option with a strike price of 96.56438 on a bond that pays off 102.5 at time 3.0 years. Equation (28.20) shows that the value of the first option is 0.009085 and the value of the second option is 0.806143. The total value of the option is therefore 0.815238.

Problem 28.10.

Put-call parity shows that:[3]

$$c + I + PV(K) = p + B_0$$

or

$$p = c + PV(K) - (B_0 - I)$$

[2] Problem 28.8 should refer to Problem 28.7 (not 23.7). There is a typo in the first printing of the book.
[3] Problem 28.10 should refer to Problem 28.9 (not 23.9). There is a typo in the first printing of the book.

where c is the call price, K is the strike price, I is the present value of the coupons, and B_0 is the bond price. In this case $c = 0.8152$, $PV(K) = 99 \times P(0,2.1) = 87.1222$, $B_0 - I = 2.5 \times P(0,2.5) + 102.5 \times P(0,3) = 87.4730$ so that the put price is

$$0.8152 + 87.1222 - 87.4730 = 0.4644$$

Problem 28.11.

Using the notation in the text $P(0,T) = e^{-0.1\times 1} = 0.9048$ and $P(0,s) = e^{-0.1\times 5} = 0.6065$. Also

$$\sigma_P = \frac{0.01}{0.08}(1 - e^{-4\times 0.08})\sqrt{\frac{1 - e^{-2\times 0.08\times 1}}{2\times 0.08}} = 0.0329$$

and $h = -0.4192$ so that the call price is

$$100 \times 0.6065N(h) - 68 \times 0.9048N(h - \sigma_P) = 0.439$$

Problem 28.12.

The relevant parameters for the Hull–White model are $a = 0.05$ and $\sigma = 0.015$. Setting $\Delta t = 0.4$

$$\hat{B}(2.1,3) = \frac{B(2.1,3)}{B(2.1,2.5)} \times 0.4 = 0.88888$$

Also from equation (28.26), $\hat{A}(2.1,3) = 0.99925$ The first stage is to calculate the value of R at time 2.1 years which is such that the value of the bond at that time is 99. Denoting this value of R by R^*, we must solve

$$2.5e^{-R^*\times 0.4} + 102.5\hat{A}(2.1,3)e^{-\hat{B}(2.1,3)R^*} = 99$$

The solution to this for R^* turns out to be 6.626%. The option on the coupon bond is decomposed into an option with a strike price of 96.565 on a zero-coupon bond with a principal of 102.5 and an option with a strike price of 2.435 on a zero-coupon bond with a principal of 2.5. The first option is worth 0.0105 and the second option is worth 0.9341. The total value of the option is therefore 0.9446.

Problem 28.13.

We will consider instantaneous forward and futures rates. (A more general result involving the forward and futures rate applying to a period of time between T_1 and T_2 is proved in Technical Note 1 on the author's site.)

Because $P(t,T) = A(t,T)e^{-r(T-t)}$ the process for $P(t,T)$ is from Itô's lemma

$$dP(t,T) = \ldots - \sigma(T-t)P(t,T)dz$$

Define $F(t,T)$ as the instantaneous forward rate for maturity T. The process for $F(0,T)$ is from Itô's lemma

$$dF(0,T) = \ldots + \sigma dz$$

The instantaneous forward rate with maturity T has a drift of zero in a world that is forward risk neutral with respect to $P(t,T)$. This is a world where the market price of risk is $-\sigma(T-t)$. When we move to a world where the market price of risk is zero the drift of the forward rate increases to $\sigma^2(T-t)$. Integrating this between $t=0$ and $t=T$ wee see that the forward rate grows by a total of $\sigma^2T^2/2$ between time 0 and time T in a world where the market price of risk is zero. The futures price has zero growth rate in this world. At time T the forward price equals the futures price. It follows that the futures price must exceed the forward price by $\sigma^2T^2/2$ at time zero. This is consistent with the formula in Section 6.4.

Define $F(0,t)$ and $G(0,t)$ as the instantaneous forward and futures rate for maturity t so that

$$G(0,t)-F(0,t)=\sigma^2t^2/2$$

and

$$G_t(0,t)-F_t(0,t)=\sigma^2t$$

In the traditional risk-neutral world the expected value of r at time t is the futures rate, $G(0,t)$. This means that the expected growth in r at time t must be $G_t(0,t)$ so that $\theta(t)=G_t(0,t)$. It follows that

$$\theta(t)=F_t(0,t)+\sigma^2t$$

This is equation (28.11).

Problem 28.14.

In this case we have $P(t,T)=A(t,T)e^{-B(t,T)r}$ so that from Itô's lemma

$$dP(t,T)=\ldots-\sigma B(t,T)P(t,T)dz$$

Define $F(t,T)$ as the instantaneous forward rate for maturity T. The process for $F(0,T)$ is from Itô's lemma

$$dF(0,T)=\ldots+\sigma e^{-a(T-t)}dz$$

This has drift of zero in a world that is forward risk neutral with respect to $P(t,T)$. This is a world where the market price of risk is $-\sigma B(t,T)$. When we move to a world where the market price of risk is zero the drift of $F(0,T)$ increases to $\sigma^2e^{-a(T-t)}B(t,T)$. Integrating this between $t=0$ and $t=T$ wee see that the forward rate grows by a total of

$$\frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

between time 0 and time T in a world where the market price of risk is zero. The futures price has zero growth rate in this world. At time T the forward price equals the futures price. It follows that the futures price must exceed the forward price by

$$\frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

at time zero.[4]

Define $F(0,t)$ and $G(0,t)$ as the instantaneous forward and futures rate for maturity t so that

$$G(0,t)-F(0,t)=\frac{\sigma^2}{2a^2}(1-e^{-at})^2$$

and

$$G_t(0,t)-F_t(0,t)=\frac{\sigma^2}{a}(1-e^{-at})e^{-at}$$

In the traditional risk-neutral world the expected value of r at time t is the futures rate, $G(0,t)$. This means that the expected growth in r at time t must be $G_t(0,t)-a[r-G(0,t)]$ so that $\theta(t)-ar=G_t(0,t)-a[r-G(0,t)]$. It follows that

$$\theta(t)=G_t(0,t)+aG(0,t)]$$

$$=F_t(0,t)+aF(0,t)+\frac{\sigma^2}{a}(1-e^{-at})e^{-at}+\frac{\sigma^2}{2a}(1-e^{-at})^2$$

$$=F_t(0,t)+aF(0,t)+\frac{\sigma^2}{2a}(1-e^{-2at})$$

This proves equation (28.14).

Problem 28.15.

The time step, Δt, is 1 so that $\Delta r=0.015\sqrt{3}=0.02598$. Also $j_{\max}=4$ showing that the branching method should change four steps from the center of the tree. With only three steps we never reach the point where the branching changes. The tree is shown in Figure S28.1.

[4]To produce a result relating the futures rate for the period between times T_1 and T_2 to the forward rate between this period we can proceed as in Technical Note 1 on the author's web site. The drift of the forward rate is

$$\frac{\sigma^2B(t,T_2)^2-\sigma^2B(t,T_1)^2}{2(T_2-T_1)}$$

$$=\frac{\sigma^2}{2a^2(T_2-T_1)}[e^{at}(-2e^{-aT_2}+2e^{-aT_1})+e^{2at}[e^{-2aT_2}-e^{-2aT_1}]$$

Integrating between time 0 and time T_1 we get

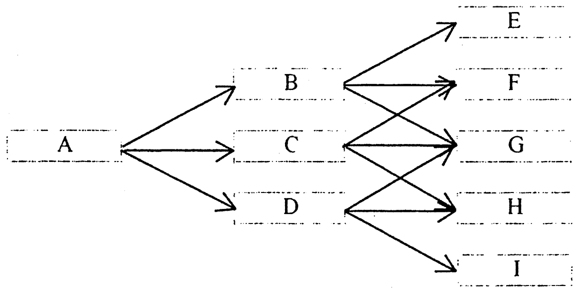
$$\frac{\sigma^2}{2a^2(T_2-T_1)}[(e^{aT_1}-1)(-2e^{-aT_2}+2e^{-aT_1})/a+(e^{2aT_1}-1)(e^{-2aT_2}-e^{-2aT_1})/(2a)]$$

$$=\frac{\sigma^2B(T_1,T_2)}{4a^2(T_2-T_1)}[4(1-e^{-aT_1})-(1-e^{-2aT_1})(1+e^{a(T_2-T_1)})]$$

$$=\frac{B(T_1,T_2)}{T_2-T_1}[B(T_1,T_2)(1-e^{-2aT_1})+2aB(0,T_1)^2]\frac{\sigma^2}{4a}$$

This is the amount by which the futures price exceeds the forward price at time zero.

Figure S28.1: Tree for Problem 28.15.



Node	A	B	C	D	E	F	G	H	I
r	10.00%	12.61%	10.01%	7.41%	15.24%	12.64%	10.04%	7.44%	4.84%
p_u	0.1667	0.1429	0.1667	0.1929	0.1217	0.1429	0.1667	0.1929	0.2217
p_m	0.6666	0.6642	0.6666	0.6642	0.6567	0.6642	0.6666	0.6642	0.6567
p_d	0.1667	0.1929	0.1667	0.1429	0.2217	0.1929	0.1667	0.1429	0.1217

Problem 28.16.

A two-year zero-coupon bond pays off $100 at the ends of the final branches. At node B it is worth $100e^{-0.12\times 1} = 88.69$. At node C it is worth $100e^{-0.10\times 1} = 90.48$. At node D it is worth $100e^{-0.08\times 1} = 92.31$. It follows that at node A the bond is worth

$$(88.69 \times 0.25 + 90.48 \times 0.5 + 92.31 \times 0.25)e^{-0.1\times 1} = 81.88$$

or $81.88.

Problem 28.17.

A two-year zero-coupon bond pays off $100 at time two years. At node B it is worth $100e^{-0.0693\times 1} = 93.30$. At node C it is worth $100e^{-0.0520\times 1} = 94.93$. At node D it is worth $100e^{-0.0347\times 1} = 96.59$. It follows that at node A the bond is worth

$$(93.30 \times 0.167 + 94.93 \times 0.666 + 96.59 \times 0.167)e^{-0.0382\times 1} = 91.37$$

or $91.37. Because $91.37 = 100e^{-0.04512\times 2}$, the price of the two-year bond agrees with the initial term structure.

Problem 28.18.

An 18-month zero-coupon bond pays off $100 at the final nodes of the tree. At node E it is worth $100e^{-0.088\times 0.5} = 95.70$. At node F it is worth $100e^{-0.0648\times 0.5} = 96.81$. At node G it is worth $100e^{-0.0477\times0.5} = 97.64$. At node H it is worth $100e^{-0.0351\times0.5} = 98.26$. At node I it is worth $100e^{0.0259\times0.5} = 98.71$. At node B it is worth

$$(0.118\times95.70+0.654\times96.81+0.228\times97.64)e^{-0.0564\times0.5} = 94.17$$

Similarly at nodes C and D it is worth 95.60 and 96.68. The value at node A is therefore

$$(0.167\times94.17+0.666\times95.60+0.167\times96.68)e^{-0.0343\times0.5} = 93.92$$

The 18-month zero rate is $0.08-0.05e^{-0.18\times1.5} = 0.0418$. This gives the price of the 18-month zero-coupon bond as $100e^{-0.0418\times1.5} = 93.92$ showing that the tree agrees with the initial term structure.

Problem 28.19.

The calibration of a one-factor interest rate model involves determining its volatility parameters so that it matches the market prices of actively traded interest rate options as closely as possible.

Problem 28.20.

The option prices are 0.1302, 0.0814, 0.0580, and 0.0274. The implied Black volatilities are 14.28%, 13.64%, 13.24%, and 12.81%.

Problem 28.21.

From equation (28.15)

$$P(t,t+\Delta t) = A(t,t+\Delta t)e^{-r(t)B(t,t+\Delta t)}$$

Also

$$P(t,t+\Delta t) = e^{-R(t)\Delta t}$$

so that

$$e^{-R(t)\Delta t} = A(t,t+\Delta t)e^{-r(t)B(t,t+\Delta t)}$$

or

$$e^{-r(t)B(t,T)} = \frac{e^{-R(t)B(t,T)\Delta t/B(t,t+\Delta t)}}{A(t,t+\Delta t)^{B(t,T)/B(t,t+\Delta t)}}$$

Hence equation (28.25) is true with

$$\hat{B}(t,T) = \frac{B(t,T)\Delta t}{B(t,t+\Delta t}$$

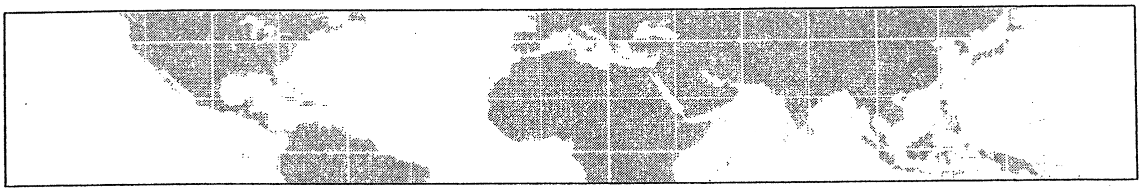
and

$$\hat{A}(t,T) = \frac{A(t,T)}{A(t,t+\Delta t)^{B(t,T)/B(t,t+\Delta t)}}$$

or

$$\ln\hat{A}(t,T) = \ln A(t,T) - \frac{B(t,T)}{B(t,t+\Delta t)}\ln A(t,t+\Delta t)$$

Substituting for $\ln A(t,T)$ and $\ln A(t,t+\Delta t)$ we obtain equation (28.26).



Chapter 29

Interest Rate Derivatives: HJM and LMM

SOLUTIONS TO QUESTIONS AND PROBLEMS

Problem 29.1.

In a Markov model the expected change and volatility of the short rate at time t depend only on the value of the short rate at time t. In a non-Markov model they depend on the history of the short rate prior to time t.

Problem 29.2.

Equation (29.1) becomes

$$dP(t,T) = r(t)P(t,T)\,dt + \sum_k v_k(t,T,\Omega_t)P(t,T)\,dz_k(t)$$

so that

$$d\ln[P(t,T_1)] = \left[r(t) - \sum_k \frac{v_k(t,T_1,\Omega_t)^2}{2}\right] dt + \sum_k v_k(t,T_1,\Omega_t)\,dz_k(t)$$

and

$$d\ln[P(t,T_2)] = \left[r(t) - \sum_k \frac{v_k(t,T_2,\Omega_t)^2}{2}\right] dt + v_k(t,T_2,\Omega_t)\,dz_k(t)$$

From equation (29.2)

$$df(t,T_1,T_2) = \frac{\sum_k [v_k(t,T_2,\Omega_t)^2 - v_k(t,T_1,\Omega_t)^2]}{2(T_2-T_1)}\,dt + \sum_k \frac{v_k(t,T_1,\Omega_t) - v_k(t,T_2,\Omega_t)}{T_2-T_1}\,dz_k(t)$$

Putting $T_1 = T$ and $T_2 = T + \Delta t$ and taking limits as Δt tends to zero this becomes

$$dF(t,T) = \sum_k \left[v_k(t,T,\Omega_t)\frac{\partial v_k(t,T,\Omega_t)}{\partial T}\right] dt - \sum_k \left[\frac{\partial v_k(t,T,\Omega_t)}{\partial T}\right] dz_k(t)$$

Using $v_k(t,t,\Omega_t) = 0$

$$v_k(t,T,\Omega_t) = \int_t^T \frac{\partial v_k(t,\tau,\Omega_t)}{\partial \tau}\,d\tau$$

The result in equation (29.6) follows by substituting

$$s_k(t,T,\Omega_t) = \frac{\partial v_k(t,T,\Omega_t)}{\partial T}$$

Problem 29.3.

Using the notation in Section 29.1, when s is constant,[5]

$$v_T(t,T) = s \qquad v_{TT}(t,T) = 0$$

Integrating $v_T(t,T)$

$$v(t,T) = sT + \alpha(t)$$

for some function α. Using the fact that $v(T,T) = 0$, we must have

$$v(t,T) = s(T-t)$$

Using the notation from Chapter 28, in Ho–Lee $P(t,T) = A(t,T)e^{-r(T-t)}$. The standard deviation of the short rate is constant. It follows from Itô's lemma that the standard deviation of the bond price is a constant times the bond price times $T-t$. The volatility of the bond price is therefore a constant times $T-t$. This shows that Ho–Lee is consistent with a constant s.

Problem 29.4.

Using the notation in Section 29.1, when $v_T(t,T) = s(t,T) = \sigma e^{-a(T-t)}$ 4

$$v_{TT}(t,T) = -a\sigma e^{-a(T-t)}$$

Integrating $v_T(t,T)$

$$v(t,T) = -\frac{1}{a}\sigma e^{-a(T-t)} + \alpha(t)$$

for some function α. Using the fact that $v(T,T) = 0$, we must have

$$v(t,T) = \frac{\sigma}{a}[1 - e^{-a(T-t)}] = \sigma B(t,T)$$

Using the notation from Chapter 28, in Hull–White $P(t,T) = A(t,T)e^{-rB(t,T)}$. The standard deviation of the short rate is constant, σ. It follows from Itô's lemma that the standard deviation of the bond price is $\sigma P(t,T)B(t,T)$. The volatility of the bond price is therefore $\sigma B(t,T)$. This shows that Hull-White is consistent with $s(t,T) = \sigma e^{-a(T-t)}$.

Problem 29.5.

LMM is a similar model to HJM.[6] It has the advantage over HJM that it involves forward rates that are readily observable. HJM involves instantaneous forward rates.

[5] In the first printing of this book the references to LMM should be to HJM.
[6] In the first printing of this book the references to BGM should be to HJM.

Problem 29.6.

A ratchet cap tends to provide relatively low payoffs if a high (low) interest rate at one reset date is followed by a high (low) interest rate at the next reset date. High payoffs occur when a low interest rate is followed by a high interest rate. As the number of factors increase, the correlation between successive forward rates declines and there is a greater chance that a low interest rate will be followed by a high interest rate.

Problem 29.7.

Equation (29.10) can be written

$$dF_k(t) = \zeta_k(t)F_k(t)\sum_{i=m(t)}^{k}\frac{\delta_i F_i(t)\zeta_i(t)}{1+\delta_i F_i(t)}dt + \zeta_k(t)F_k(t)\,dz$$

As δ_i tends to zero, $\zeta_i(t)F_i(t)$ becomes the standard deviation of the instantaneous t_i-maturity forward rate at time t. Using the notation of Section 29.1 this is $s(t,t_i,\Omega_t)$. As δ_i tends to zero

$$\sum_{i=m(t)}^{k}\frac{\delta_i F_i(t)\zeta_i(t)}{1+\delta_i F_i(t)}$$

tends to

$$\int_{\tau=t}^{t_k} s(t,\tau,\Omega_t)\,d\tau$$

Equation (29.10) therefore becomes

$$dF_k(t) = \left[s(t,t_k,\Omega_t)\int_{\tau=t}^{t_k} s(t,\tau,\Omega_t)\,d\tau\right]dt + s(t,t_k,\Omega_t)\,dz$$

This is the HJM result.

Problem 29.8.

In a ratchet cap, the cap rate equals the previous reset rate, R, plus a spread. In the notation of the text it is R_j+s. In a sticky cap the cap rate equal the previous capped rate plus a spread. In the notation of the text it is $\min(R_j,K_j)+s$. The cap rate in a ratchet cap is always at least a great as that in a sticky cap. Since the value of a cap is a decreasing function of the cap rate, it follows that a sticky cap is more expensive.

Problem 29.9.

When prepayments increase, the principal is received sooner. This increases the value of a PO. When prepayments increase, less interest is received. This decreases the value of an IO.

Problem 29.10.

A bond yield is the discount rate that causes the bond's price to equal the market price. The same discount rate is used for all maturities. An OAS is the parallel shift to the Treasury zero curve that causes the price of an instrument such as a mortgage-backed security to equal its market price.

Problem 29.11.

When there are p factors equation (29.7) becomes

$$dF_k(t) = \sum_{q=1}^{p} \zeta_{k,q}(t)F_k(t)dz_q$$

Equation (29.8) becomes

$$dF_k(t) = \sum_{q=1}^{p} \zeta_{k,q}(t)[v_{m(t),q} - v_{k+1,q}]F_k(t)dt + \sum_{q=1}^{p} \zeta_{k,q}(t)(F_k(t)dz_q$$

Equation coefficients of dz_q in

$$\ln P(t,t_i) - \ln P(t,t_{i+1}) = \ln[1+\delta_i F_i(t)]$$

Equation (29.9) therefore becomes

$$v_{i,q}(t) - v_{i+1,q}(t) = \frac{\delta_i F_i(t)\zeta_{i,q}}{1+\delta_i F_i(t)}$$

Equation (29.15) follows.

Problem 29.12.

From the equations on page 688

$$s(t) = \frac{P(t,T_0) - P(t,T_N)}{\sum_{i=0}^{N-1} \tau_i P(t,T_{i+1})}$$

and

$$\frac{P(t,T_i)}{P(t,T_0)} = \prod_{j=0}^{i-1} \frac{1}{1+\tau_j G_j(t)}$$

so that

$$s(t) = \frac{1 - \prod_{j=0}^{N-1} \frac{1}{1+\tau_j G_j(t)}}{\sum_{i=0}^{N-1} \tau_i \prod_{j=0}^{i} \frac{1}{1+\tau_j G_j(t)}}$$

(We employ the convention that empty sums equal zero and empty products equal one.) Equivalently,

$$s(t) = \frac{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)] - 1}{\sum_{i=0}^{N-1} \tau_i \prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]}$$

or

$$\ln s(t) = \ln\left\{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)] - 1\right\} - \ln\left\{\sum_{i=0}^{N-1}\tau_i \prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]\right\}$$

so that

$$\frac{1}{s(t)}\frac{\partial s(t)}{\partial G_k(t)} = \frac{\tau_k \gamma_k(t)}{1+\tau_k G_k(t)}$$

where

$$\gamma_k(t) = \frac{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)]}{\prod_{j=0}^{N-1}[1+\tau_j G_j(t)] - 1} - \frac{\sum_{i=0}^{k-1}\tau_i \prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]}{\sum_{i=0}^{N-1}\tau_i \prod_{j=i+1}^{N-1}[1+\tau_j G_j(t)]}$$

From Itô's lemma the qth component of the volatility of $s(t)$ is

$$\sum_{k=0}^{N-1}\frac{1}{s(t)}\frac{\partial s(t)}{\partial G_k(t)}\beta_{k,q}(t)G_k(t)$$

or

$$\sum_{k=0}^{N-1}\frac{\tau_k \beta_{k,q}(t)G_k(t)\gamma_k(t)}{1+\tau_k G_k(t)}$$

The variance rate of $s(t)$ is therefore

$$V(t) = \sum_{q=1}^{p}\left[\sum_{k=0}^{N-1}\frac{\tau_k \beta_{k,q}(t)G_k(t)\gamma_k(t)}{1+\tau_k G_k(t)}\right]^2$$

Problem 29.13.

$$1+\tau_j G_j(t) = \prod_{m=1}^{M}[1+\tau_{j,m}G_{j,m}(t)]$$

so that

$$\ln[1+\tau_j G_j(t)] = \sum_{m=1}^{M}\ln[1+\tau_{j,m}G_{j,m}(t)]$$

Equating coefficients of dz_q

$$\frac{\tau_j \beta_{j,q}(t)G_j(t)}{1+\tau_j G_j(t)} = \sum_{m=1}^{M}\frac{\tau_{j,m}\beta_{j,m,q}(t)G_{j,m}(t)}{1+\tau_{j,m}G_{j,m}(t)}$$

If we assume that $G_{j,m}(t) = G_{j,m}(0)$ for the purposes of calculating the swap volatility we see from equation (29.17) that the volatility becomes

$$\sqrt{\frac{1}{T_0}\int_{t=0}^{T_0}\sum_{q=1}^{p}\left[\sum_{k=n}^{N-1}\sum_{m=1}^{M}\frac{\tau_{k,m}\beta_{k,m,q}(t)G_{k,m}(0)\gamma_k(0)}{1+\tau_{k,m}G_{k,m}(0)}\right]^2 dt}$$

This is equation (29.19).

Chapter 30

Swaps Revisited

SOLUTIONS TO QUESTIONS AND PROBLEMS

Problem 30.1.

The target payment dates are July 11, 2004; January 11, 2005; July 11, 2005; January 11, 2006; July 11, 2006; January 11, 2007; July 11, 2007; January 11, 2008; July 11, 2008; January 11, 2009. These occur on Sunday, Tuesday, Monday, Wednesday, Tuesday, Thursday, Wednesday, Friday, Friday, and Sunday respectively with no holidays. The actual payment dates are therefore July 12, 2004; January 11, 2005; July 11, 2005; January 11, 2006; July 11, 2006; January 11, 2007; July 11, 2007; January 11, 2008; July 11, 2008; January 12, 2009. The fixed rate day count convention is Actual/365. There are 182 days between January 11, 2004 and July 11, 2004. This means that the fixed payments on July 11, 2004 is

$$\frac{182}{365} \times 0.06 \times 100,000,000 = \$2,991,781$$

Similarly subsequent fixed cash flows are: \$3,024,658, \$2,975,342, \$3,024,658, \$2,975,342, \$3,024,658, \$2,975,342 \$3,024,658, \$2,991,781 \$3,024,658.

Problem 30.2.

Yes. The swap is the same as one on twice the principal where half the fixed rate is exchanged for the LIBOR rate.

Problem 30.3.

The final fixed payment is in millions of dollars:

$$[(4 \times 1.0415 + 4) \times 1.0415 + 4] \times 1.0415 + 4 = 17.0238$$

The final floating payment assuming forward rates are realized is

$$[(4.05 \times 1.041 + 4.05) \times 1.041 + 4.05] \times 1.041 + 4.05 = 17.2238$$

The value of the swap is therefore $-0.2000/(1.04^4) = -0.1710$ or –\$171,000.

Problem 30.4.

The value is zero. The receive side is the same as the pay side with the cash flows compounded forward at LIBOR. Compounding cash flows forward at LIBOR does not change their value.

Problem 30.5.

In theory, a new floating-for-floating swap should involve exchanging LIBOR in one currency for LIBOR in another currency (with no spreads added). In practice, macroeconomic effects give rise to spreads. Financial institutions often adjust the discount rates they use to allow for this. Suppose that USD LIBOR is always exchanged Swiss franc LIBOR plus 15 basis points. Financial institutions would discount USD cash flows at USD LIBOR and Swiss franc cash flows at LIBOR plus 15 basis points. This would ensure that the floating-for-floating swap is valued consistently with the market.

Problem 30.6.

In this case $y_i = 0.05$, $\sigma_{y,i} = 0.13$, $\tau_i = 0.5$, $F_i = 0.05$, $\sigma_{F,i} = 0.18$, and $\rho_i = 0.7$ for all i. It is still true that $G_i'(y_i) = -437.603$ and $G_i''(y_i) = 2261.23$. Equation (30.2) gives the total convexity/timing adjustment as $0.0000892t_i$ or 0.892 basis points per year until the swap rate is observed. The swap rate in three years should be assumed to be 5.0268%. The value of the swap is $119,069.

Problem 30.7.

In a plain vanilla swap we can enter into a series of FRAs to exchange the floating cash flows for their values if the "assume forward rates are realized rule" is used. In the case of a compounding swap Section 30.2 shows that we are able to enter into a series of FRAs that exchange the final floating rate cash flow for its value when the "assume forward rates are realized rule" is used. There is no way of entering into FRAs so that the floating-rate cash flows in a LIBOR-in-arrears swap are exchanged for their values when the "assume forward rates are realized rule" is used.

Problem 30.8.

Suppose that the fixed rate accrues only when the floating reference rate is below R_X and above R_Y where $R_Y < R_X$. In this case the swap is a regular swap plus two series of binary options, one for each day of the life of the swap. Using the notation in the text, the risk-neutral probability that LIBOR will be above R_X on day i is $N(d_2)$ where

$$d_2 = \frac{\ln(F_i/R_X) - \sigma_i^2 t_i^2/2}{\sigma_i\sqrt{t_i}}$$

The probability that it will be below R_Y where $R_Y < R_X$ is $N(-d_2')$ where

$$d_2' = \frac{\ln(F_i/R_Y) - \sigma_i^2 t_i^2/2}{\sigma_i\sqrt{t_i}}$$

From the viewpoint of the party paying fixed, the swap is a regular swap plus binary options. The binary options corresponding to day i have a total value of

$$\frac{QL}{n_2}P(0,s_i)[N(d_2)+N(-d_2')]$$

(This ignores the small timing adjustment mentioned in Section 30.6.)