8

Measuring the Risk

8.1 Risk Measures

In this section we present some traditional risk measures based on the present value formula used in the markets for the quoting of prices and yields to maturity (*ytms*). These measures are calculated by trading software in order to at least partially manage the risk in instruments and portfolios.

8.1.1 Delta

The delta value of an instrument shows the sensitivity of the price[1] to changes in the main source of risk of an underlying instrument. Examples of sources of risk are yield curves changes and the price of underlying asset and the delta is calculated separately for these.

8.1.1.1 Delta Price

The price delta calculations are only applicable for derivative instruments with an underlying instrument (that have a price[2]), valued on the basis of a non-term structure model. It shows the change in theoretical price given a unit change in the price of the underlying.

[1] With price, we here refer to the present value sometimes called the fair value of a financial instrument.
[2] Interest rate is not a tradeble instrument. But a bond option have an underlying instrument, the bond.

J.R.M. Röman, *Analytical Finance: Volume II*,
DOI 10.1007/978-3-319-52584-6_8

The general *Delta Price* formula is

$$\Delta_{price} = \frac{\partial (PV)}{\partial U} = \frac{PV(U+h) - PV(U)}{h} \times scale$$

where U is the current value of the main source of risk and h is the differentiation step. When the market price of the underlying is used, the price shift is a relative shift, that is,

$$h = 0.0001 \times U$$

When a theoretical underlying price is used, the price shift is an absolute shift, that is,

$$h = 0.01$$

Sometime this value is called **delta explicit**.

Example 8.1.1.1

We want to calculate the *Delta Price* for a bond using

a) the market price of the underlying, and
b) the theoretical price of the underlying.

First, assume that the current market price of the underlying asset is $U = 100.17$. The present value will now be calculated twice, the first time using the current price of the underlying and the second time after applying a shift to the price of the underlying with the shift size expressed as $h = U\text{x } 0.0001 = 0.010017$.

We obtain

$$\Delta_{price} = \frac{PV(U+h) - PV(U)}{h} \cdot \frac{100}{Nom}$$

where *Nom* is the nominal amount, typically one million.

Next, assume that the theoretical price of the underlying asset is $U = 100.15$. The shift size is now $h = 0.01$ and $U + h = 100.16$. The present value is calculated twice, using these two different prices for the underlying, which gives the result with the previous formula.

8.1.1.2 Delta Yield

The yield curve delta shows the change in the present value, given a shift of 1 basis point (bp) in all yield curves used. The shift is applied to the annually compounded zero coupon curve, using the day count fraction Act/365.

The yield delta can either refer to an upward or a downward shift of yields. The general **Delta Yield** formula is

$$\Delta_{yield} = \left[PV\left(\underline{r} + h\right) - PV\left(\underline{r}\right)\right] * scale$$

where

$$h = \pm 0.00001 \quad \text{and} \quad scale = 1000.$$

Sometimes, but not always, the shift step used in the calculations is thus actually 1/1000 of a bp to get high accuracy of the slope, but the result is scaled to a 1 bp shift. The delta can also be broken down according to different time buckets, to illustrate the sensitivity to a particular shift in a given time bucket. These time buckets can be defined in the software used to calculate the risk. This can be 1, 2, 3, 7 days followed by 2 and 4 weeks, then 3, 6, 9 and 12 months and 2, 3, 4, 5, 7, 10, 12, 15, 20, 25 and 30 years. In such a way that the time buckets are well defined between the given terms.

Example 8.1.1.2

We will calculate the *Delta Yield* of a bond using the theoretical price of the underlying interest rate, the yield y.

The calculations are based on the present value using the current yield curves and the present value using yield curves that are shifted with a shift size of 1×10^{-5}. The result is then scaled so that the shift represents a size of one bp. We have

$$\Delta_{yield} = \left[PV(\underline{y} + 0.00001) - PV(\underline{y}) \right] \cdot 1000$$

8.1.2 Duration and Convexity

The Macaulay duration (or just duration) is a measure of the price sensitivity of an interest rate instrument with the respect to an absolute change in the *ytm*. This measure can be interpreted as the average life of the bond, when a bond is the financial instrument. It is easy to show that the duration for a zero-coupon bond is the same as its time to maturity.

The *modified duration* measures the percentage bond price change for an absolute yield change. It can also be interpreted as the negative slope of the price-yield relation. In a similar way convexity can be interpreted as the curvature of the relation between the price and *ytm*.

We use the following 4 risk measures

- Macaulay's duration
- Modified duration
- Dollar duration
- Convexity

They will depend on

- Time to maturity
- The coupon rate
- The coupon frequency
- The market rate

For a bond, the duration measure of the weighted average of the times until the fixed cash flows are received and is given in year. Therefore, the duration of a zero-coupon bond is the same as its time to maturity. A coupon-paying bond has duration less than its time to maturity because part of the cash flows, the coupons, are paid before maturity. This will be illustrated later.

Suppose we have n cash flows $c_i, i = 1, 2, \ldots, n$ at times t_i. Then, the quoted bond price, P is given by (using continuous compounding):

$$P = \sum_{i=1}^{n} c_i e^{-yt_i}$$

where y is the ytm. The duration is defined as:

$$D = \frac{1}{P} \sum_{i=1}^{n} t_i c_i e^{-yt_i} = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-yt_i}}{P} \right]$$

where the factor in [.] is the present value of each cash flow using the continuously compounded ytm for discounting of the cash flows. This can also be expressed as

$$D = -\frac{1+y}{P}\frac{\partial P}{\partial y} = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{t_i \cdot C_i}{(1+y)^{t_i}} + \frac{t_m \cdot N}{(1+y)^{t_m}} \right].$$

where for simplicity we assume there is 1 coupon per year and

P = the present value (which is the quoted price, if this exist),
y = the bond ytm,
C = coupon size (the coupon rate times the nominal amount, N)
N = the nominal amount (or the principal)
n = number of years to maturity (if we have 1 coupon per year).

P is given by

$$P = \frac{N}{(1+y)^n} + \sum_{i=1}^{n} \frac{C}{(1+y)^i}$$

The factor $(1 + y)$ comes from the fact that duration is defined as the derivative with respect to the ytm in the market quoting formula. For continuously compounded *ytm* we get

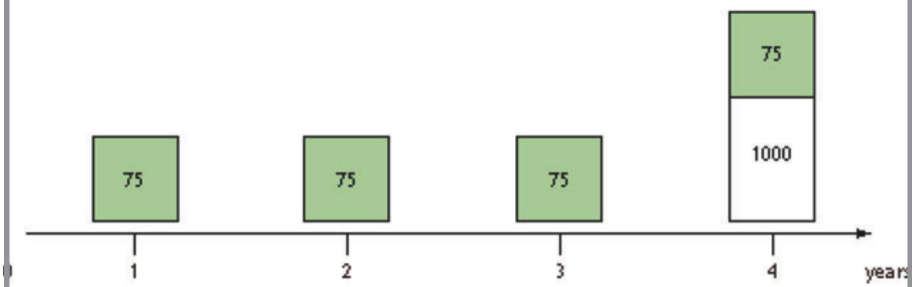
$$D = -\frac{1}{P}\frac{\partial P}{\partial y} = -\frac{1}{P}\frac{\partial P}{\partial y}\left[\sum_{i=1}^{n} C_i \cdot e^{-y \cdot t_i} + N \cdot e^{-y \cdot t_m}\right]$$

$$= \frac{1}{P}\left[\sum_{i=1}^{n} C_i \cdot t_i \cdot e^{-y \cdot t_i} + N \cdot t_m \cdot e^{-y \cdot t_m}\right].$$

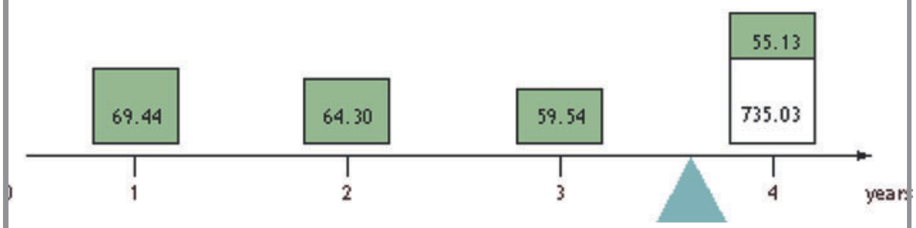
If the coupon frequency is m times per year, the formulas has to be slightly modified.

Example 8.1.2.3

Given a 4-year maturity bond with a principal 1000 and an annual coupon rate of 7.5%. This bond will have the following projected[3] cash flows.



Suppose we have a constant (flat) market rate of 8%. Then the present value of the cash flows will be



We then get duration of 3.6 year.

8.1.2.1 Swap Duration

Duration, as we have seen for aforementioned bonds, can also be defined for other kinds of interest rate instruments. Portfolio managers like to find the duration for their entire portfolio. Therefore we also

[3] Projected cashflows are the coupons payes or received in the future given by the coupon rate.

need also to define a duration for other instruments. One problem is that no *ytm* is defined for other instruments.

Some managers use an approximation for swaps by calculating the duration of the fixed leg as 0.75 times the time to maturity. Similarly, one may calculate the duration of the floating leg as 0.5 times the tenor. This means that for a floating leg with 3-month tenor, the duration should be $0.25 * 0.5 = 0.125$ years, for a 6-month tenor $0.5 * 0.5 = 0.25$ and for a 1-year tenor it will be 0.5.

Example 8.1.2.4

A 10-year receiver swap with a 3-month tenor will have a duration of $7.5 - 0.125 = 7.375$ year. Similarly, the payer swap has -7.375 year. Portfolio managers used payer swaptions to hedge duration from bonds.

A better calculation of the Swap duration would be to use the interest rate sensitivity and use the following formula

$$Dur_{swap} = \frac{MV_0 - MV_1}{N + MV_0} \times 10000$$

Here MV_0 is the market value of the swap and MV_1 the market value we get if we shift the market swap curve 1 bp (up) and N is the nominal amount.

Example 8.1.2.5

In the following table, we show how the duration varies for a semi-annual coupon-paying bond when the *ytm* is 5% and the coupon rate is 1, 2, 5 and 10%, respectively.

	Coupon Rate			
Years to maturity	1%	2%	5%	10%
1	0.997	0,995	0.988	0.977
2	1.984	1.969	1.928	1.868
5	4.875	4.763	4.485	4.156
10	9.416	8.950	7.989	7.107
25	20.164	17.715	14.536	12.754
50	26,666	22.284	18.765	17.384
100	22.572	21.200	20.363	20.067
Infinity	20.500	20.500	20.500	20.500

When time to maturity increases to the limit, we find the value

$$D \xrightarrow[T \to \infty]{} \frac{1 + y/f}{y}$$

Where y is the *ytm* per annum.

Example 8.1.2.6
A 3-year bond with principal 1000 paying an annual coupon of 10%. If the market price of this bond is 951.97 with a yield of 12%, the duration is given by

$$D = \frac{0.10 \cdot 1000/(1+0.12) + 2 \cdot 0.10 \cdot 1000/(1+0.12)^2 + 3 \cdot (1+0.10) \cdot 1000/(1+0.12)^3}{951.97}$$
$$= 2.73 \text{ years}$$

If we have a portfolio of interest rate instruments, the **portfolio duration** is defined by

$$D_{portfolio} = \frac{1}{PV_{portfolio}} \cdot \sum_i PV_i \cdot D_i$$

8.1.3 Modified Duration, Dollar Duration and DV01

In contrast to the Macaulay duration, modified duration (*MD*) is a price sensitivity measure, defined as the percentage derivative of price with respect to yield. MD applies when a bond or other asset price is considered as a function of yield. In this case one can measure the logarithmic derivative with respect to yield. The MD shows the change in price in percentage terms, resulting from a change in the *ytm*. It is defined by

$$MD = -\frac{1}{P}\frac{\partial P}{\partial y} = \{\text{using the simple formula}\} = \frac{D}{1 + y/n}$$

where n is the number of cash flows per year and D is the Macaulay Duration:

$$D = \frac{1}{P}\left\{\sum_{i=1}^{n} \frac{t_i \cdot C_i}{(1+y)^{t_i}} + \frac{t_n \cdot N}{(1+y)^{t_n}}\right\}.$$

The duration gives a value of the risk. Long duration ⇔ high risk.

Definition 8.1.3.1. *Dollar duration* (DV01) measures the change in price (in money, £, $, SEK) if the market interest rate increases with 1%.

$$DD = MD \cdot N$$

The **DV01** is defined as the derivative of the value with respect to yield.

$$D_{\$} = DV01 = -\frac{\partial PV(y)}{\partial y}$$

DV01 is analogous to the delta in derivative pricing since it is the ratio of a price change in output (dollars) to a unit change in input (1 bp of yield). DV01 is called Dollar duration because it is the change in price in *dollars,* not in *percentages.* It gives the dollar variation in a bond's value per unit change in the yield. It is often measured per 1 bp – DV01 is short for "dollar value of a 01" (or 1 bp).

DV01 can be used for instruments with zero upfront value such as interest rate swaps where percentage changes and MD are less useful.

For a par bond and a flat yield curve, the DV01 is the derivative of the price with respect to the yield, and PV01, the value of a one-dollar annuity will actually have the same value.

8.1.3.1 PV01 – Val01 – BPV

The names PV01 (or Val01, present value of a bp) refers to the change in the present value on a shift of 1 bp (1/100 of a %) on the yield curve. Often, this is also referred as a BPV (the bp value). PV01 also refers to the value of a 1 dollar or 1 bp annuity.

Definition 8.1.3.2. The *Base Point Value* measures the change in price if the market rate increases by 1 bp (1bp = 0.01%).

$$BPV = \frac{D_{modified}(\%)}{100} \cdot \frac{DirtyPrice}{100}$$

$Val01$ is calculated as

$$Val01 = P(YTM - 0.5bp) - P(YTM + 0.5bp)$$

where P represents the dirty price and bp 1 bp. The shifts are added to the yield compounded according to the period of the bond.

In the BPV formula we first divide the MD by 100 to convert it from a percentage into a decimal (i.e. 5% is 0.05). The second divisor of 100 reduces the scale of risk from a 100 bp change in yield (MD) to just 1 bp.

Example 8.1.3.7

Calculation of a Base-Point Value (*BPV*) – its price sensitivity to a 1 bp change in yield?

Security:	5% US Treasury note
Type:	Semi-annual, actual/actual
Price	99.48
Accued interest:	0.84
MD:	1.50 %

$$BPV = \frac{1.50}{100} \cdot \frac{99.48 + 0.84}{100} = 0.015048$$

Thus, a 1 bp rise in the bond's yield will result in:

- A fall in the price from 95.4800 to 95.4654
- A loss of USD 1.46 cents per USD 100 nominal
- A loss of USD 145.98 on a USD 1 million position

BPVs tend to come out as very small figures with many decimal places. For convenience, many bond analysis systems scale the BPV figure by a factor of 100, so in our example the reported **Risk Factor** would be 1.4598. Thus, a 100-point change in the bond's yield would result in:

- A fall in the price from 95.48 to 94.02 (minus 1.46)
- A loss of USD 1.46 per 100 nominal
- A loss of USD 14,598 on a USD 1 million position

8.1.3.2 CV01

CV01 is the sensitivity to a 1bp shift in credit spreads.

8.1.4 Convexity

Convexity measures the percental change in the MD if the market rate increases with 1 bp. This can also be defined as the change in BPV for a change in the yield. The **convexity** can be calculated as the derivative of the duration with respect to the yield or as the second order derivative of the bond price with respect to time. This is the corresponding measure to gamma in option theory.

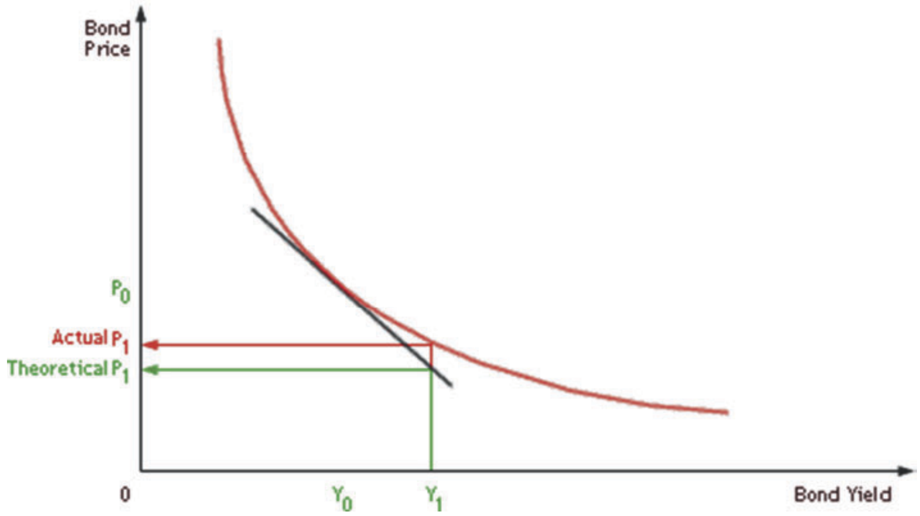


Fig. 8.1

The convexity is a nonlinear function, which can be compared by gamma in the option analysis. In Fig. 8.1, we show the error in the theoretical price if we do not consider the convexity on a change in yield.

If we take the derivative of the bond price with respect to the yield (continuously compounded) we get

$$\frac{\partial P}{\partial y} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i} = -PD$$

That is,

$$\frac{\Delta P}{P} = -D\Delta y$$

This can be applied to a portfolio as well. If we express y in terms of annual profit we get

$$\Delta P = -\frac{P \cdot D \cdot \Delta y}{1 + y/m} = -P \cdot MD \cdot \Delta y$$

where m is the number of annual coupons. The convexity can be written as

$$Cnvx = \frac{1}{2P}\sum_{i=1}^{n} t_i^2 c_i e^{-yt_i}$$

or

$$Cnvx = \frac{(1+y)^2}{P}\frac{\partial^2 P}{\partial y^2} = \frac{1}{2P}\left[\sum_{i=1}^{n}\frac{t_i \cdot (t_i+1)\cdot C_i}{(1+y)^{t_i}} + \frac{t_m \cdot (t_m+1)\cdot N}{(1+y)^{t_m}}\right]$$

Example 8.1.4.8

Consider a 7% bond with semi-annual coupons 3 years to maturity. Assume that the bond is selling at a yield of 8%. We then have

Year	Cashflow	Discounting	PV	PV/Price	Yeai*pv/Price
0.5	3.5	0.962	3.365	0.035	0.017
1.0	3.5	0.925	3.236	0.033	0.033
1.5	3.5	0.889	3.111	0.032	0.048
2.0	3.5	0.855	2.992	0.031	0.061
2.5	3.5	0.822	2.877	0.030	0.074
3.0	103.5	0.790	81.798	0.840	2.520

The price is the sum of individual PV, giving 97.379. The duration is 2.753. Suppose the yield changes to 8.2%, then the change in bond price is approximated by:

$$\frac{1}{P}\frac{\Delta P}{\Delta y} \approx -\frac{Dur}{1+y} \quad \Rightarrow \quad \Delta P = -\frac{97.379 \cdot 0.2\% \cdot 2.753}{1+0.04} = -0.5156$$

Using the convexity the change in the bond price on a change in yield is given by:

$$\frac{\Delta P}{P} \cong -\frac{Dur}{1+y}\Delta y + \frac{1}{2}Cnvx \cdot (\Delta y)^2$$

So, if X and Y are 2 different portfolios with the same duration, the difference in convexity can become large for changes in the yield. The convexity, C has its maximum close to a coupon payment day.

8.1.5 Gamma

The gamma value shows the extent of the change in the delta value when the same shift is applied to the delta as was used when the delta was first calculated.

8.1.5.1 Gamma Price

The *Gamma price* differentiation formula is

$$\Gamma_{price} = \frac{\partial\,(U+h) - \partial\,(U)}{h} = \frac{PV\,(U+2h) - 2PV\,(U+h) + PV\,(U)}{h^2} * scale.$$

The *Gamma price* calculates the change in *Delta Price*, given a unit change in the underlying asset price. This value is sometimes called **gamma explicit**.

Example 8.1.5.9
We are going to calculate the *Gamma Price* of a bond call option where the price of the underlying asset is $U = 100.17$, the shift step $h = U * 0.0001 = 0.010017$ and the present values are $PV(U) = 6,419.56, PV(U+h) = 6,434,49$ *and* $PV(U+2h) = 6,449.46$. The *Gamma Price* is then given by:

$$\Gamma_{price} = \frac{6,449.46 - 2 * 6,434,49 + 6,419.56}{0.010017^2} \cdot \frac{100}{1,000,000} = 0.0399.$$

8.1.5.2 Gamma Yield

The *Gamma yield* formula can be represented as

$$\Gamma_{yield} = \left[\partial\left(\underline{y}+h\right) - \partial\left(\underline{y}\right)\right] = \left[PV\left(\underline{y}+2h\right) - 2PV\left(\underline{y}+h\right) + PV\left(\underline{y}\right)\right] * scale$$

The yield curve gamma can, like the delta, be broken down into different time buckets. If this is the case, the gamma value shows the change in delta for the corresponding time bucket given a 1 bp change in the yield curve as a whole and not just in the individual bucket.

Example 8.1.5.10
Calculate the *Gamma yield* of a call option on a bond where the shift size is 0.00001 and the present values are $PV(y) = 7,377.6943211, PV(y+h) = 7,377.69432195$ and $PV(y+2h) = 7,377.6943231$:

$$\Gamma_{yield} = \left[PV(y+2h) - 2PV(y+h) + PV(y)\right] * scale$$
$$= [7,377.6943231 - 2 * 7,377.69432195 + 7,377.6943211] \cdot 1,000^2 = 0.3.$$

8.1.6 Accrued Interest

Accrued interest is defined and calculated as the upcoming coupon payment times the number of days after the previous coupon was paid using the relevant day-count convention. When expressed in percentage points of the nominal amount of the bond it is equal to the difference between the dirty price and the (quoted) clean price.

8.1.7 Rho

Rho represents the change in the present value, given a shift of 1 bp in the repo curve. In the calculations, the yield is normally shifted by 1/1000 of a bp. The result is then scaled to a 1 bp shift by multiplying it by 1000.

Example 8.1.7.11

To calculate the *Rho* value of a put option on a bond we base the calculations on the present value of the option with unchanged conditions and the present value calculated using a repo curve that is shifted 1 bp

$$\rho = \left[PV(r_{repo} + 0.00001) - PV(r_{repo})\right] \cdot 1000$$

8.1.8 Theta

The *Theta* value shows the change in present value (*PV*) from the valuation date until the next calendar date, given unchanged market conditions.

Unchanged market conditions here imply that the yield curve will stay the same on both dates. For generic periods, the zero coupon rates are the same on both days, while all rates for fixed dates will be rolled down the curve by 1 day. Forward rates for fixed periods will also be affected when shifting the zero coupon yield curve.

Volatility values used for option pricing are also affected by a shift in the valuation date. This is only significant when a volatility landscape with a slope in the option expiry dimension is used. When shifting the valuation date, the time to expiration of the option will be 1 day shorter and another volatility will be fetched. When the underlying market price is used in the calculations, the underlying price is not affected by the 1-day forward shift.

> **Example 8.1.8.12**
> Consider a bond position in a 5-year government bond.
>
> $$\Theta = PV(2004-11-05) - PV(2004-11-06)$$
>
> where *PV*(2004-11-05) and *PV*(2004-11-06) are calculated after moving the valuation date and the yield curve rates one day forward. In a positive interest rate environment, the *Theta* of a bond position is normally positive.

The theta value has 2 components

- The first is due to the decrease in time to maturity. When valuing the bond 1 day later, the value of the bond will be higher because of the shorter time used when discounting the cash flows.
- The second component is due to the shape of the yield curve. If the yield curve is upwards sloping, each cash flow will be discounted with a slightly lower yield when valued as of tomorrow.

This is often referred to as "rolling down the curve". The exact slope of the curve will decide the size and sign of the contribution to the theta value. If the yield curve has a negative slope, this contribution can make the theta value negative for a bond.

8.1.8.1 Theta Classic

The *Theta Classic* value shows the change in *PV* from the valuation date until the next calendar date, given unchanged market conditions.

Here, an unchanged market condition means that the yield curve for tomorrow will be the one implied by today's forwards. There is no "rolling-down-the-curve" effect.

Volatility values, repo rates and underlying prices used for option valuation are kept constant for the 2 days in the calculations. The underlying price is kept constant, even if the theoretical price of the underlying is used.

For options priced with the Black-Scholes formula, the *Theta Classic* value represents the time value derived from that formula.

8.1.9 Vega

The *Vega* value shows the change in the *PV* from an upward shift in volatility of 1 %. The calculations are normally performed using a shift size of 0.01% and then scaling the result to a 1% shift:

$$\nu = [PV(\sigma + 0.01) - PV(\sigma)] \cdot 100$$

8.1.10 YTM

As we have seen, for bond price quoting, several *ytm* calculation methods are available.

As the ISMA and the Moosmüller methods were presented earlier, we will not repeat the formulas here. When calculating *ytm*, bond coupons are treated as follows

- Coupons are estimated to be the full yearly coupon dividend divided by the number of coupons.
- The time factor used when discounting each cash flow is:
- Time to next coupon according to instruments day count convention + (number of Coupon x number of coupons per year)

A simple version of the *ytm* formula looks like this

$$P = \sum_i \frac{c_i}{(1 + ytm)^{t_i}} + \frac{100}{(1 + ytm)^{t_n}}$$

where c_i are the coupons of the bond, t_i the time for the payouts and P the market price of the bond. With continuously compounding *ytms*, we can write this formula as

$$P = \sum_i c_i \cdot e^{-t_i \cdot ytm} + 100 \cdot e^{-t_n \cdot ytm}$$

In the case of promissory loans, a minor correction has to be made because the accrued interest is not paid on the value date, but deducted from the next coupon. However, this simple pricing formula is not used very frequently in practice, because it is a cumbersome process to incorporate the exact time elements of the coupons. If the adjustment to the coupon payment dates, to account for non-banking days are ignored, the formula can be simplified. Since all bonds pay coupons periodically, time can be measured in coupon periods defined previously.

The main problem with the use of *ytm* as a measure of interest rates is that it is not consistent across instruments. One 5-year bond will typically have a different *ytm* compared with another 5-year bond if they have different coupons. It is therefore impossible to associate a single interest rate with each maturity. One way of overcoming this problem is to use forward-rates.

Forward-rates are interest rates that are assumed to apply over a given periods between 2 future times. This contrasts with yields that are assumed to apply up to maturity, with a different yield for each bond. This is why the forward rate can be calculated by an arbitrage condition and the spot rate. The forward rate will also depend on the method used for the rate, if using continuously compounding rate or using a certain day-count method.

8.1.10.1 Simple Yield Formula

Another formula used for transformations between price and *ytm* in fixed income markets is the *Simple yield-to-maturity formula*, also known as *Japanese yield*. It takes into account the effect of the Capital gain or loss on maturity of the bond, as well as the current yield. Any Capital appreciation/depreciation is assumed to occur uniformly over the bond's life

$$ytm = \frac{c + \frac{Nom - P_{clean}}{L}}{P_{clean}},$$

where c is the annual coupon rate in % and L the life to maturity in years. A special day count fraction is used: L = the number of days to maturity, excluding February 29 in any year divided by 365.

8.1.10.2 The Money Market Formula

The money market formula is given by

$$P_{dirty} = \frac{Nom + c}{1 + ytm \cdot T}$$

where c is the annual coupon rate in % and T the time from spot to maturity in years, using the day count method of the instrument.

This method is relevant for instruments with one remaining coupon and for non-coupon instruments (zero bonds and bills). For the latter the ytm reduces to the simple annualized rate of interest.

8.1.11 Portfolio Immunization Using Duration and Convexity

Let $PV_L(y)$ denote the present value of some liability for a given yield y and $PV_j(y)$ the present value of some bonds, j in a bond portfolio given the yield y. We suppose in the following example that we have 3 bonds (i.e. $j = 1, 2, 3$). The present value of the bond portfolio is then given by:

$$PV_p(y) = PV_1(y) + PV_2(y) + PV_3(y)$$

Furthermore let $D_L(y)$ denote the MD of the liability, $D_j(y)$ the MD of bond j, $C_L(y)$ the convexity of the liability and $C_j(y)$ the convexity of bond j.

The derivative is then given by

$$\frac{d}{dy} PV_L(y) = -D_L(y) \cdot PV_L(y)$$

$$\frac{d}{dy} PV_p(y) = -\left[D_1(y) \cdot PV_1(y) + D_2(y) \cdot PV_2(y) + D_3(y) \cdot PV_3(y)\right]$$

and

$$\frac{d^2}{dy^2} PV_p(y) = \left[C_1(y) \cdot PV_1(y) + C_2(y) \cdot PV_2(y) + C_3(y) \cdot PV_3(y)\right]$$

for all y. Ideally, we would like to have $PV_p(y) = PV_L(y)$ for all y, since that would immunize the liability using the bond portfolio. If y changes, the portfolio can still be used to meet the liability.

Say that y_1 is the present yield value. Certainly we want $PV_p(y_1) = PV_L(y_1)$. We can conclude that PV_p is a better approximation to PV_L at y_1 if also $PV_p'(y_1) = PV_L'(y_1)$ and $PV_p''(y_1) = PV_L''(y_1)$.

Thus, we want to achieve

$$\begin{cases} PV_p(y_1) = PV_L(y_1) \\ PV_p'(y_1) = PV_L'(y_1) \\ PV_p''(y_1) = PV_L''(y_1) \end{cases}$$

If we use a Taylor series expansion

$$PV_L(y) = PV_L(y_1) + PV_L'(y_1)(y - y_1) + \frac{1}{2}PV_L''(y_1)(y - y_1)^2 + \text{higher order terms}$$

where the right hand side converges and represents the function $PV_L(y)$ for those values of y for which all the derivatives exist and for which the higher order terms go to zero. Note that

$$PV_L(y) - PV_L(y_1) = PV_L'(y_1)(y - y_1) + \frac{1}{2}PV_L''(y_1)(y - y_1)^2 + \text{higher order terms}$$

If we let $\Delta y = y - y_1$, we get

$$PV_L(y) - PV_L(y_1) \approx PV_L'(y_1) \cdot \Delta y + \frac{1}{2}PV_L''(y_1)\Delta y^2$$

when we drop the higher order terms.

If we have $PV_p'(y_1 = PV_L'(y_1)$ and $PV_p''(y_1) = PV_L''(y_1)$, we can conclude that $PV_L(y) - PV_L(y_1) \approx PV_p(y) - PV_p(y_1)$. Thus, a change in the value of the portfolio would track the change in the value of the liability if there were a change in the yield.

We are essentially trying to construct $PV_p(y)$ to make it a good approximation to $PV_L(y)$. We can make the approximation perfect for $y = y_1$, and "good" for y "near" y_1. The higher order terms we consider the better the approximation we get. To have $PV_p(y_1) = PV_L(y_1), PV_p'(y_1) = PV_L'(y_1)$ and $PV_p''(y_1) = PV_L''(y_1)$ we need

$$\begin{cases} PV_1(y_1) + PV_2(y_1) + PV_3(y_1) = PV_L(y_1) \\ PV_1(y_1) \cdot D_1(y_1) + PV_2(y_1) \cdot D_2(y_1) + PV_3(y_1) \cdot D_3(y_1) = PV_L(y_1) \cdot D_L(y_1) \\ PV_1(y_1) \cdot C_1(y_1) + PV_2(y_1) \cdot C_2(y_1) + PV_3(y_1) \cdot C_3(y_1) = PV_L(y_1) \cdot C_L(y_1) \end{cases}$$

If these three equations hold then PV_p is a good approximation to PV_L up to second order.

Because $PV_j(y_1)$ is the present value of bond j in the portfolio at y_1 we can choose it. It is just the amount of bond j we purchase. Further, the duration values and convexity values do not depend upon the amounts of the bonds that we purchase; they just depend on the yield. Hence, we have a system of three equations in three unknowns to solve. The unknowns are the $PV_j(y_1)$.

Note we can use durations instead of MDs because all the durations have the same modifier and it cancels out.

8.1.12 The Fisher-Weil Duration and Convexity

The Macaulay duration measures do not provide any information for how the price of a bond is affected by a change in the zero-coupon yield curve. Therefore, they are not useful for comparing the interest rate risk of different bonds. The problem is that the Macaulay measures are defined in terms of the bond's own *ytm*, and a given change in the zero-coupon yield curve will generally result in different changes in the yields of different bonds. It is easy to show that the changes in the yields of all bonds will be the same if and only if the zero-coupon yield curve is always flat. In particular, the yield curve is only allowed to move in parallel shifts. Such an assumption is not only unrealistic, it also conflicts with the no-arbitrage principle.

In 1938 Macaulay defined an alternative duration measure based on the zero-coupon yield curve rather than the bond's own yield. After decades of neglect this duration measure, it was revived by Fisher and Weil in 1971. They demonstrated the relevance of the measure for constructing immunization strategies. We will refer to this duration measure as the *Fisher-Weil duration.* The precise definition is

$$D^{FW} = \frac{1}{P}\sum_{i=1}^{n} t_i c_i e^{-y_i t_i} = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-y_i t_i}}{P}\right]$$

or

$$D^{FW} = \frac{1}{P}\cdot\left[\frac{N\cdot t_n}{(1+y_n)^n} + \sum_{i=1}^{n}\frac{C\cdot t_i}{(1+y_i)^i}\right]$$

where y_i is the zero-coupon yield prevailing at time 0 for the period up to time t_i. If the changes in all the zero-coupon yields are identical, the relative price change is proportional to the Fisher-Weil duration. Consequently, the Fisher-Weil duration represents the price sensitivity towards infinitesimal parallel shifts of the zero-coupon yield curve. Note that an infinitesimal parallel shift of the curve of continuously compounded yields corresponds to an infinitesimal proportional shift in the curve of yearly compounded yields. We can also define the *Fisher-Weil convexity* as

$$Cnvx^{FW} = \frac{1}{2P} \sum_{i=1}^{n} t_i^2 c_i e^{-y_i t_i}$$

or

$$Cnvx^{FW} = \frac{1}{2P} \cdot \left[\frac{N \cdot t_n^2}{(1 + y_n)^n} + \sum_{i=1}^{n} \frac{C \cdot t_i^2}{(1 + y_i)^i} \right]$$

8.1.13 Hedging with Duration

Suppose we want to use a futures contract to hedge a position in an interest rate instrument. Let F_c be price of a futures contract, D_F the duration of the underlying instrument, P the future value of the portfolio we want to hedge and D_P the duration on the portfolio at the end day of the contract. We then have

$$\Delta P = -P \cdot D_P \cdot \Delta y$$
$$\Delta F_c \approx -F_c \cdot D_F \cdot \Delta y$$

The duration based hedge factor expressed in the number of futures contracts is given by

$$N = \frac{P \cdot D_P}{F_c \cdot D_F}$$

This approach is subject to the following complications

- We have to guess which instrument that is cheapest to deliver (CTD) at expiration.
- The CTD instrument may change in time.

- The convexity.
- We can have non-parallel shifts on the yield curve.

8.1.14 Shifting the Zero-Coupon Yield Curve

The delta and gamma yield values are calculated by shifting the segment of the zero coupon yield curve that corresponds to the time bucket. The bucket shifts are constructed so that their sum always represents a single bp shift in the whole curve.

The calculations used when shifting the yield curve are actually performed using a differentiation step of 1/1000 bps and then scaling to 1 bp by multiplying by 1000. The reason for this is that changes in implied forward rate calculations can be quite substantial when shifting a segment of the curve in which only one of the forward points is located. These substantial changes can then suddenly make out-of-the money options, caps and floors, for example, in the money. This kind of non-linear effect should be avoided in a first order measure such as delta.

The choice of yield shift method affects the distribution of risk figures (such as delta and gamma) between time buckets. Their sum, that is, the total risk figure, is however not affected.

8.1.14.1 Rectangle Shift

The yield curve is shifted 1 bp between the end of the previous bucket specification (exclusive) and the end of the current bucket specification (inclusive) (Fig. 8.2).

8.1.14.2 Triangle Shift

The shift in the yield curve takes the form of a triangle with its apex at the bucket date and ending at the boundaries of the two adjacent buckets. In the first and last bucket, the two triangles are extended indefinitely to ensure that the sum of all the shifts corresponds to a total parallel shift of 1 bp (Fig. 8.3).

Example 8.1.14.13

Suppose there are only four time buckets: 3y, 5y, 10y and the rest bucket. The 5y bucket has its maximum shift (1 bp) at 5y, and linearly decreasing to 0 at 3y and to 10y. The shift at 7y is 0.6 bp.

The 10y bucket has its maximum shift at 10y and decrease linearly to 5 years. The shift at 7y is 0.4 bp. It also has a linearly decreasing shift towards higher maturities; with 0 shifts at 15y (a date determined using the step between 5y and 10y). The shift at 12y is 0.6 bp.

Therefore, the Rest Bucket has a 1 bp shift starting at 15y, running parallel to the end of time. It is linearly decreasing from 15y back to 10y. With this definition, the total of all buckets will add up to a parallel shift of 1 bp, which is important.

It may be argued that a rest bucket should not be used at all in the application, since it introduces the seemingly strange 15y time point. The reason is that if we use rectangular shifts, the rest bucket is needed for including all sensitivities above 10y. The 10y bucket includes every maturity between 5y and 10 y (including 10y).

8.1.14.3 Smooth Shift

The shift in the yield curve takes the form of a smooth shape with its highest point at the bucket date and ending at the boundaries of the two adjacent buckets.

The smooth yield shift method is recommended for contracts that are valued according to finite difference methods. The smooth shape implies a continuous shape of the yield curve, which is essential when using the finite difference solver.

The smooth shift has a cubic formula, where the first half representing the upward slope is defined by

$$3 \cdot t^2 - 2 \cdot t^3$$

where t is the time factor of the bucket ($0 < t < 1$). The second half of the shift is symmetrical to the first half.(Fig. 8.4)

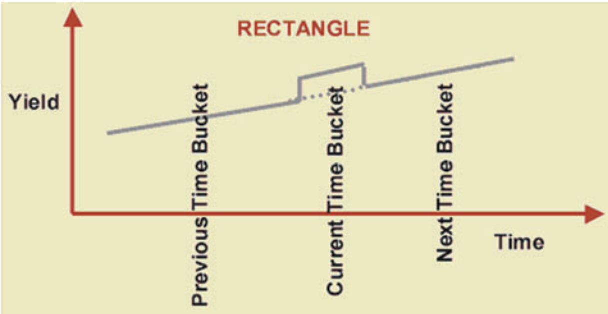


Fig. 8.2 A rectangular shift on the yield curve

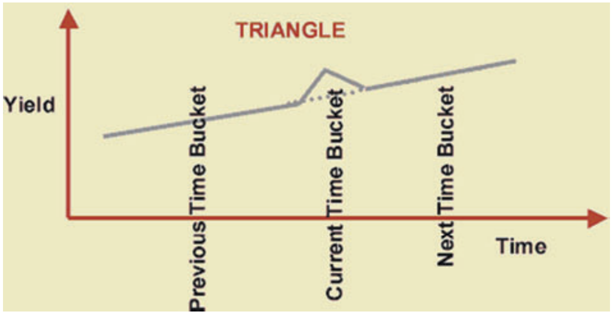


Fig. 8.3 A triangular shift on the yield curve

Fig. 8.4 A smooth shift on the yield curve