5

Yield Curves

5.1 Introduction to Yield Curves

Ordering the current spot yields to maturities for any group of bonds. By maturity we get a so-called yield curve. This curve is often represented as a graph with time to maturity on the horizontal axis and yields on the vertical axis. The group is usually defined as bonds by the same issuer and/or the same credit rating. Thus, we speak of yield curves for government bonds, for mortgage bonds or for corporate bonds of the same credit rating. The word bond here is used in the academic sense which means bills, notes and bonds. Interest rates in international or domestic time deposit markets too can be ordered by maturity and credit class. Thus, we get London inter-bank offered rate (LIBOR) or XIBOR yield curves or yield curves for domestic deposits in any currency. There are many different yield curves. In general, yield curves may slope upwards or downwards, their shapes can be concave, convex or have humps.

In Table 5.1 and Fig. 5.1, we show the yield curve for UK government bonds. This data was taken from *The Financial Times* September 6, 2016.

Today, many countries have negative interest rates. A few years ago, many actors (if not all) thought that negative interest rates could not exist. But things have changed and nowadays it is a fact. In Table 5.2 we have market prices for the Swedish government securities[1] (bills and bonds) at 2016-09-09.

[1] https://www.riksgalden.se/sv/For-investerare/Statspapper/Utestaende-statspapper/

J.R.M. Röman, *Analytical Finance: Volume II*,
DOI 10.1007/978-3-319-52584-6_5

Table 5.1 Government bond yields in UK 2016-09-06

Maturity	Yield	Today'schange	1 week ago	1 month ago
1 Month	0.18%	0	0.18%	0.20%
3 Month	0.23%	> −0.01	0.22%	0.27%
6 Month	0.31%	0.05	0.31%	0.28%
1 Year	0.15%	0.06	0.17%	0.16%
2 Year	0.10%	< 0.01	0.16%	0.14%
3 Year	0.09%	0	0.14%	0.13%
4 Year	0.16%	> −0.01	0.19%	0.17%
5 Year	0.19%	> −0.01	0.22%	0.21%
7 Year	0.36%	0	0.36%	0.43%
8 Year	0.46%	> −0.01	0.44%	0.57%
9 Year	0.56%	−0.06	0.54%	0.59%
10 Year	0.66%	> −0.01	0.64%	0.67%
15 Year	1.03%	> −0.01	0.96%	1.14%
20 Year	1.18%	> −0.01	1.11%	1.32%
30 Year	1.30%	> −0.01	1.24%	1.49%

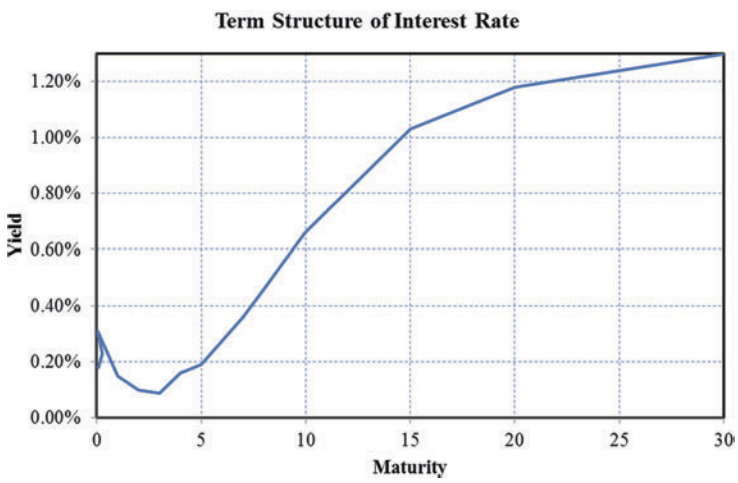

Fig. 5.1 Government bond yields in UK 2016-09-06

The bootstrapped zero-coupon rates from Table 5.2 are shown in Fig. 5.2. As we can see, the zero rate is negative up to nine years.

Term structure models are based on the assumption that the whole term structure of interest rates can be derived from the stochastic behaviour of one or many variables. The reason for modelling the entire term structure is to make all model prices internally consistent.

Table 5.2 Quotes of Swedish Government securities[2]

Securities	Issue date	Maturity	Coupon	Price
STB 21 Sep 16	2016-03-11	2016-09-21	0	100.02
STB 19 Oct 16	2016-07-01	2016-10-19	0	100.09
STB 16 Nov 16	2016-08-05	2016-11-16	0	100.15
STB 21 Dec 16	2016-06-03	2016-12-21	0	100.23
STB 15 Mar 17	2016-09-02	2017-03-15	0	100.42
SGB 1051 3.75% 12 Aug 17	2006-09-15	2017-08-12	3.75	104.45
SGB 1052 4.25% 12 Mar 19	2007-11-21	2019-03-12	4.25	114.47
SGB 1047 5% 1 Dec 20	2004-01-28	2020-12-01	5.00	127.47
SGB 1054 3.5% 1 Jun 22	2011-02-09	2022-06-01	3.50	123.18
SGB 1057 1.5% 13 Nov 23	2012-10-22	2023-11-13	1.50	113.68
SGB 1058 2.5% 12 May 25	2014-02-03	2025-05-12	2.50	123.27
SGB 1059 1.0% 12 Nov 26	2015-05-22	2026-11-12	1.00	109.57
SGB 1056 2.25% 1 Jun 32	2012-03-20	2032-06-01	2.25	124.84
SGB 1053 3.5% 30 Mar 39	2009-03-30	2039-03-30	3.50	153.64

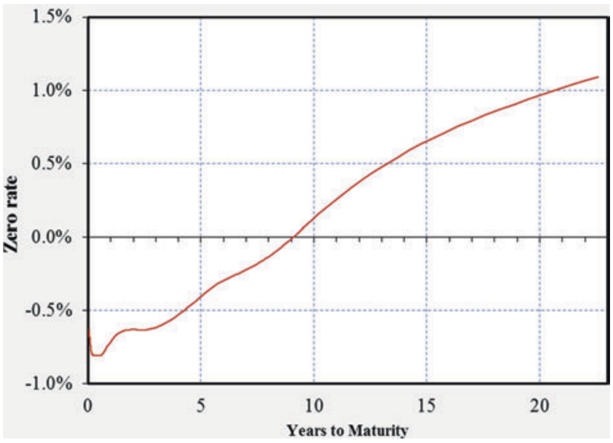

Fig. 5.2

In categorising these models, two properties are significant:

- **Number of state variables**
  - Most models lack analytical solutions and have to be solved using numerical methods. The computing time increases dramatically for each new state variable.
- **External consistency**.
  - By external consistency, we mean coherence between the model term structure and the observed term structure. When the model

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[2] Source, https://www.riksgalden.se/sv/For-investerare/Statspapper/Utestaende-statspapper/

is used to price derivative instruments, it is essential that the underlying instrument is priced in accordance with observed market prices.

Although this yield curve obviously includes T-bills, T-notes and T-bonds, we refer to these collectively as "bonds" in the usual academic sense. In parallel to the US government yield curve, there are yield curves for domestic time deposits between banks, for the international deposit market (LIBOR), for interest rate swaps, and for municipal and corporate bonds. Closest to the US government yield curve are curves for instruments with the highest credit rankings.

As an example consider the yield curve on the same day for USD interest rate swaps. The yields are given in buckets as seen next:

2010-05-10	1	0.5069
2010-05-11	2	0.5323
2010-05-12	3	0.5235
2010-05-16	7	0.4882
2010-06-08	30	0.3082
2010-08-07	90	0.2225
2010-11-05	180	0.2944
2011-02-03	270	0.4067
2011-05-09	365	0.5564
2012-05-08	730	1.1271
2013-05-08	1095	1.6642
2014-05-08	1460	2.0844
2015-05-08	1825	2.4084
2017-05-07	2555	2.8126
2019-05-07	3285	3.1287
2020-05-06	3650	3.2642
2025-05-05	5475	3.4964
2030-05-04	7300	3.4964
2040-05-01	10950	3.4964

This data uses a day count conversion 30/360.

To find the yield to maturity (*ytm*) for intermediate maturities, interpolation is used. In C/C++ the following function can be used to interpolate the *y*-values and return *y* for a given term/maturity *x* (*pX* and *pY* are arrays of length *N*),

```
double IPOL(double x, double, *pX, Double *pY, int N)
{
  if (x <= pX[0]) return pY[0];
  if (x >= pX[N-1]) return pY[N-1];
```

```
  for (int i = 1; i < N; i++)  {
    if (x == pX[i-1]) return pY[i-1];
    if (x == pX[i]) return pY[i];

    If (x > pX[i-1] && x < pX[i])
      return pY[i-1] + (x - pX[i-1])*(pY[i] - pY[i-1])/
                       (pX[i] - pX[i-1]);
  }
}
```

When you have the interpolated values, forward rates between two arbitrary future dates can be calculated.

```
double forwardRate(int Days1, double r_t1, int Days2, double r_t2)
{
  return pow((pow(1.0 + r_t2, Days2/365.0))/
             (pow(1.0 + r_t1, Days1/365.0)), 365.0/
             (Days2 - Days1)) - 1.0;
}
```

Both these examples show yield curves that are upwards sloping. This is the typical case. Why is this so? Why do *ytms* for instruments of the same credit rating differ because of maturity? The traditional explanations are:

- Expectations theory
- Liquidity preference theory
- Market segmentation theory

Briefly, the expectations theory argues that the slope of the yield curve (or equivalently the term premium) reflects the market's average expectations about future interest rates/yields. Lending short term while borrowing long term you can lock in yields on any forward starting loan today. This approach was used to evaluate interest rate swaps in the preceding chapter. In particular, it was argued that implied forward rates calculated from the current yield curve were unbiased forecasts of future spot rates, while the liquidity preference theory argues that forward rates are always biased high because investors prefer liquidity. Market segmentation based on credit ratings is clearly an empirical fact but it has also been used to explain why yield curves typically should slope upwards. The reason is that there was a chronic shortage of long-term investors in relation to the supply. Typically insurance companies prefer to invest their cash long term while banks rely more heavily on short-term funding.

5.1.1 Credit Ratings

Next, we see the notation used for credit rating *Moody's* and *Standard and Poor's*:

Standard and Poor's		Moody's
AAA		Aaa
AA		Aa
A	Investment grade	A
BBB		Baa
BB		Ba
B	Speculative grade	B
CCC	"Junk"	Caa
CC		Ca
C		C
D	Default	D

A few years ago, government bonds issued by the government of Argentina was classed as D.

Some models use a transition matrix to describe the probabilities for transitions between the different credit ratings. For the financial crises in 2008, see Table 5.3. The transition matrix is an implied matrix sampled from bond prices in emerging markets. It describes the transition probabilities between different credit ratings. By the same method, it is also possible to calculate a cumulative default probability matrix, which is illustrated in Table 5.4.

Table 5.3 The Transition Matrix in the beginning of 2008

1-Year Transition Matrix								
From	To Rating							
Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	88,658%	10,294%	1,017%	0,000%	0,031%	0,000%	0,000%	0,000%
AA	1,079%	88,705%	9,553%	0,342%	0,145%	0,145%	0,000%	0,031%
A	0,063%	2,876%	90,205%	5,919%	0,740%	0,177%	0,010%	0,010%
BBB	0,053%	0,339%	7,069%	85,238%	6,053%	1,005%	0,085%	0,159%
BB	0,033%	0,077%	0,557%	5,680%	83,572%	8,083%	0,535%	1,464%
B	0,011%	0,044%	0,174%	0,652%	6,595%	82,703%	2,760%	7,062%
CCC	0,000%	0,000%	0,660%	1,050%	3,050%	6,110%	62,970%	26,160%
D	0,000%	0,000%	0,000%	0,000%	0,000%	0,000%	0,000%	100,000%

Table 5.4 The cumulative default probability matrix

Cumulative Default Probability Matrix							
	AAA	AA	A	BBB	BB	B	CCC
1 Year	0,00%	0,31%	0,1%	0,159%	1,464 %	7,062%	26160%
2 Years	0,004%	0,073%	0,056%	0,477%	347%	13722%	43111%
3 Years	0,12%	0,127%	0,145%	0,950%	5678%	19828%	54255%
4 Years	0,027%	0,198%	0,284%	1,568%	8,157%	25,339%	6172%
5 Years							

5.2 Zero-coupon Yield Curves

An important subclass among the yield curves are so-called zero-coupon yield curves. These can be derived almost directly from money market instruments or calculated from interest rate swaps or groups of coupon paying bonds using special bootstrapping techniques. A zero-coupon yield curve defines a set of discount factors that can be used for the discounting of future cash flows. An older name for this construct was the term structure of interest rates.

While the previous examples showed *ytm*s for each bond individually, the zero-coupon yield curve shows a curve that when used for the discounting of the future cash flows of all the bonds in the curve replicate their market prices. Note that the yield curves in the examples were obtained by applying the present value (*PV*) formula to each individual bond in order to translate from price to *ytm*. Thus, different *ytm*s were being used for "discounting" cash flows at future dates depending on the bond. This is inconsistent. The proper way to discount future cash payments is to apply the same *ytm* to all the bonds which pay cash on the same future date. All the underlying cash flows from the whole set of bonds should be discounted with a unique yield that only depends on the future date of the cash flow. There should only be one yield per future date. Otherwise portfolios of bonds could be constructed that exploits any mispriced cash flow. This is a no arbitrage requirement.

So when the quoted *ytms* are given for a subset of bonds, we need to use a method called bootstrapping to calculate a matching zero-coupon yield curve. With this technique, we strip the bonds to create virtual zero-coupon bonds from the coupons and the principal. This is not a trivial exercise and the results will be different depending on the method used. One method is used by the US Treasury Department and the results are published as Separate Trading of Registered Interest and Principal of Securities (STRIPS).

The coupons and principal of normal bonds are split up, creating artificial zero-coupon bonds of longer maturity than would otherwise be available.

Example 5.2.0.1

Let us study the price of a bond maturing in exactly one year with semi-annual coupons of 10 % and a quoted *ytm* of 5.951%. Using the *PV* formula to derive the cash price, we get

$$\frac{5}{1+\frac{0.05951}{2}} + \frac{105}{\left(1+\frac{0.05951}{2}\right)^2} = 103.874$$

This is not necessarily the one-year, zero-coupon yield. Suppose the six-months zero-coupon yield is 4%. Then the matching zero-coupon yield for one year, say s, must be given by

$$\frac{5}{1+\frac{0.04}{2}} + \frac{105}{\left(1+\frac{s}{2}\right)^2} = 103.874$$

Solving the equation we get $s = 6.0\%$ which is close but not exactly equal to the given 12m yield on the coupon bond.

We know that the quoted *ytm* on a bond *y* can be used in the *PV* formula to calculate its market price. Vice versa, given the price *P*, we can find the *ytm* by solving the following equation

$$P = \sum_i \frac{c_i}{(1+ytm)^{t_i}} + \frac{100}{(1+ytm)^{t_n}}$$

where c_i is the coupons of the bond, t_i the time for the payouts and P the market price of the bond. For continuously compounded *ytms*, the formula is

$$P = \sum_i c_i \cdot e^{-t_i \cdot ytm} + 100 \cdot e^{-t_n \cdot ytm}$$

Remember that the *ytm* on a bond is only an adequate measure of the rate of return on this investment if all coupons can be reinvested at the same yield.

5.2.1 ISMA and Moosmüller

There exist a number of different methods to calculate the *ytm* and we will give two of them, International Securities Market Association (ISMA) and Moosmüller. ISMA is given by

$$\begin{aligned} P &= v^f \cdot \left[\frac{C}{H} \cdot \sum_{i=1}^{n} v^{i+1} + \left(Nom + g \cdot \frac{C}{H} \right) \cdot v^{n-1+g} \right] \\ &= v^f \cdot \left[\frac{C}{H} \cdot \frac{1 - v^n}{1 - v} + \left(Nom + g \cdot \frac{C}{H} \right) \cdot v^{n-1+g} \right] \end{aligned}$$

where

$$v = \frac{1}{1 + \frac{ytm_h}{H}}$$

ytm_h is the given H coupons per year. A common formula in Germany is the Moosmüller method that can also handle parts of coupons. The Moosmüller formula is given by

$$P = \frac{1}{1 + \frac{f \cdot ytm_h}{H}} \cdot \left[\frac{C}{H} \cdot \frac{1 - v^n}{1 - v} + \frac{Nom + g \cdot \frac{C}{H}}{1 + \frac{g \cdot ytm_h}{H}} \cdot v^{n-1+g} \right]$$

In the next section we'll show how zero-coupon yield curves can be derived from any given set of quoted market yields.