Chapter 23

Market Models

23 The LIBOR Market Model (LMM)

23.1 Introduction

The term structure models studied in the previous chapters have involved assumptions about the evolution in one or more continuously compounded interest rates, either the short rate $r(t)$ or, as in the HJM framework, the instantaneous forward rates $f(t, T)$. One drawback of these models is that they are expressed in terms of interest rates that are not directly observed in the market. Another disadvantage in HJM is that all model parameters must be recalibrated at each point in time; there is no mechanism for sequential updating. Also it is difficult to calibrate the model to actively traded prices in the market. However, many securities traded in money markets, e.g. caps, floors, swaps, and swaptions, depend on discretely compounded interest rates such as spot LIBOR rates $l(t, t + \delta)$, forward LIBOR rates $L(t, T, T + \delta)$, spot swap rates $l_{swap}(t, \delta)$ and forward swap rates $L_{swap}(t, T, \delta)$. For the pricing of these securities it seems more appropriate to apply models that are based on assumptions about the LIBOR rates directly or spot and forward swap rates.

We use the term market models for models based on assumptions about discretely compounded interest rates. Market models take the currently observed term structure of interest rates as a given and are therefore to be classified as relative pricing or pure no-arbitrage models. Consequently, they offer no insights into the determination of the current interest rates. LIBOR market models is based on assumptions about the evolution of forward LIBOR rates. Similarly, swap market models are based on assumptions about the evolution of forward swap rates. By construction, market models are not suitable for the pricing of futures and options on government bonds and similar contracts that do not depend on money market interest rates.

In the recent literature various market models have been developed, but most attention has been given to the so-called lognormal LIBOR market models. In such models the volatilities of a relevant selection of the forward LIBOR rates are assumed to be proportional to the level of the forward rate so that the distribution of the future forward LIBOR rates is lognormal. Lognormally distributed continuously compounded interest rates have unpleasant consequences, but Sandmann and Sondermann (1997) showed that models with lognormally distributed, discretely compounded rates are not subject to the same problems. Below, we will demonstrate that a lognormal assumption on the distribution of forward LIBOR rates implies that pricing formulas for caps and floors identical to Black's pricing formulas can be derived. Hence, the lognormal market models provide some support for the widespread use of Black's formula for fixed income securities.

We have to be aware that log-normal models can't handle negative rates. So we here suppose we have a market situation with strictly positive interest rates.

23.2 General LIBOR market models

In this section we will introduce a general **LIBOR market model**, also referred to, as the **BGM/J model** (Brace, Gatarek, Musiela and Jamshidian), describe some of the model's basic properties, and discuss how derivative securities can be priced within the framework of the model.

23.2.1 Model description

As was described in section 4.14, a cap is a contract that protects a floating rate borrower against paying an interest rate higher than some given rate K, the so-called cap rate. We let $T_1, ..., T_n$ denote the payment dates and assume that $T_i - T_{i-1} = \delta$ for all i. In addition we define $T_0 = T_1 - \delta$. At each time T_i ($i = 1, ..., n$) the cap gives a payoff of

$$C^i(T_i) = N\delta \max\{L(T_i, T_i - \delta) - K, 0\} = N\delta \max\{L(T_i - \delta, T_i - \delta, T_i) - K, 0\}$$

where N is the face value of the cap. A cap can be considered as a portfolio of caplets, namely one caplet for each payment date with payoffs described by the above formula.

The definition of the forward martingale measure in chapter 17 the previous section implies that the value of the above payoff can be found as the product of the expected payoff computed under the T_i-forward martingale measure and $p(t, T_i)$ the current discount factor for time T_i payments, i.e.

$$C^i(t) = N\delta\, p(t,T_i) E_t^{Q^{T_i}} \left[\max\left\{ L(T_i - \delta, T_i - \delta, T_i) - K, 0 \right\} \right]; \qquad t < T_i - \delta$$

The price of a cap can therefore be determined as the sum of the value of the caplets

$$C(t) = N\delta \sum_{i=1}^{n} p(t,T_i) E_t^{Q^{T_i}} \left[\max\left\{ L(T_i - \delta, T_i - \delta, T_i) - K, 0 \right\} \right]; \qquad t < T_0$$

For $t \geq T_0$ the first-coming payment of the cap is known so that its present value is obtained by multiplication by the risk-less discount factor, while the remaining payoffs are valued as above. The price of the corresponding floor is

$$F(t) = N\delta \sum_{i=1}^{n} p(t,T_i) E_t^{Q^{T_i}} \left[\max\left\{ K - L(T_i - \delta, T_i - \delta, T_i), 0 \right\} \right]; \qquad t < T_0$$

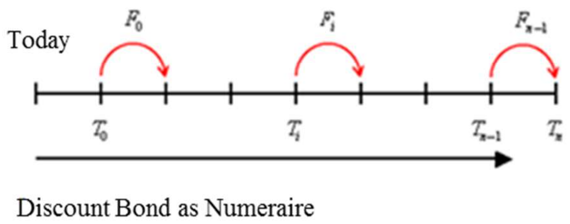
In order to compute the cap price, we need to know of the distribution of $L(T_i - \delta, T_i - \delta, T_i)$ under the T_i-forward martingale measure Q^{Ti} for each $i = 1, ..., n$. For this purpose it is natural to model the evolution of $L(t, T_i - \delta, T_i)$ under Q^{Ti} . The following argument shows that under the Q^{Ti} probability measure the drift rate of $L(t, T_i - \delta, T_i)$ is zero, i.e. $L(t, T_i - \delta, T_i)$ is a Q^{Ti}-martingale. The simpel compounded forward rate at time t spanning the future period $[T_1, T_2]$, $L(t, T_1, T_2)$ is defined by

$$\frac{p(t,T_2)}{p(t,T_1)} = \frac{1}{1 + L(t,T_1,T_2)(T_2 - T_1)}$$

We rewrite this as

$$L(t,T_i - \delta, T_i) = \frac{1}{\delta}\left(\frac{p(t,T_i - \delta)}{p(t,T_i)} - 1 \right)$$

where $\delta = T_2 - T_1$. The following diagram illustrates a set of forward rates spanning the set of dates T_i



Under the T_i-forward martingale measure Q^{Ti} the ratio between the price of any asset and the numeraire, i.e. the zero coupon bond price $p(t, T_i)$ is a martingale. In particular, the ratio $p(t, T_i - \delta)/p(t, T_i)$ is a Q^{Ti}-martingale so its expected change over any time interval is equal to zero under the Q^{Ti} measure. From the formula above it follows that the expected change (over any time interval) in the periodically compounded forward rate $L(t, T_i - \delta, T_i)$ also is zero under Q^{Ti} . We summarize the result in the following theorem

Theorem 23.1. *The forward rate $L(t, T_i - \delta, T_i)$ is a Q^{Ti}-martingale.*

Consequently, a LIBOR market model is fully specified by the number of factors (i.e. the number of standard Brownian motions) that influence the forward rates and the forward rate volatility functions. For simplicity, we focus on the one-factor models

$$dL(t,T_i-\delta,T_i)=\beta\left(t,T_i-\delta,T_i,L(t,T_j,\delta)\,|_{T_j\geq t}\right)dz(t,T_i),\quad i=1...,n.$$

where $z(t, T_i)$ is a one-dimensional standard Brownian motion under the T_i-forward martingale measure Q^{Ti}. The fourth argument in the volatility function β indicates that the volatility at time t can depend on the current values of all the modelled forward rates. In the lognormal LIBOR market models we will study later, one assumes that volatility of each forward rate is proportional to the current level of the same forward rate

$$\beta\left(t,T_i-\delta,T_i,L(t,T_j,\delta)\,|_{T_j\geq t}\right)=\gamma\left(t,T_i-\delta,T_i\right)L(t,T_i-\delta,T_i)$$

for some deterministic function γ. However, for now we continue to discuss the more general specification. We see from the general cap pricing formula that the cap price also depends on the current discount factors $p(t, T_1), p(t, T_2),\ldots, p(t, T_n)$. These discount factors can be determined by $p(t, T_0)$, and the current values of the modelled forward rates, i.e., $L(t, T_0, T_1)$, $L(t, T_1, T_2)$,.., $L(t, T_{n-1}, T_n)$,. Similarly to the HJM model, the LIBOR market models take the currently observable values of these rates as given.

23.2.2 The dynamics of all forward rates under the same probability measure

The basic specification of the LIBOR market model involves n different forward martingale measures. In order to better understand the model and to simplify the computation of some derivative prices we will describe the evolution of the relevant forward rates under a common probability measure. As discussed below, Monte Carlo simulation is often used to compute prices of certain derivatives in LIBOR market models. It is much simpler to simulate the evolution of the forward rates under a common probability measure than to simulate the evolution of each forward rate under a different martingale measure associated with the respective forward rate. One possibility is to choose one of the n different forward martingale measures used in the assumption of the model. Note that the T_i-forward martingale measure only makes sense up to time T_i. Therefore, it is appropriate to use the forward martingale measure associated with the last payment date, i.e. the T_n-forward martingale measure Q^{Tn}, since this measure applies to the entire relevant time period. In this context Q^{Tn} is sometimes referred to as *the terminal measure*. Another obvious candidate for the common probability measure is *the spot martingale measure*. Let us look at these two alternatives in more detail.

23.2.3 The terminal probability measure

We wish to describe the evolution of all modelled forward rates under a common probability measure, here the T_n-forward martingale measure. For that purpose we shall apply the following theorem that outlines how to shift between the different forward martingale measures of the LIBOR market model.

Theorem 23.2. *Assume that the evolution in the LIBOR forward rates* $L(t, T_i - \delta, T_i)$ *for* $i = 1,..., n$, *where* $T_i = T_{i-1} + \delta$, *is given by*

$$dL(t,T_i-\delta,T_i) = \beta\left(t,T_i-\delta,T_i,L(t,T_j,\delta)\,|_{T_j\geq t}\right)dz(t,T_i), \quad i = 1...,n.$$

Then the processes $z(T_i - \delta)$ *and* $z(T_i)$ *are related as follows:*

$$dz(t,T_i) = dz(t,T_i-\delta) + \frac{\delta\beta\left(t,T_i-\delta,T_i,L(t,T_j,\delta)\,|_{T_j\geq t}\right)}{1+\delta L(t,T_i-\delta,T_i)}dt$$

Using this repeatedly, we get that

$$dz(t,T_n) = dz(t,T_i) + \sum_{j=i}^{n-1}\frac{\delta\beta\left(t,T_j,T_{j+1},L(t,T_j,\delta)\,|_{T_j\geq t}\right)}{1+\delta L(t,T_j,T_{j+1})}dt$$

Consequently, for each i = 1,..., n, we can write the dynamics of $L(t, T_i - \delta, T_i)$ under the Q^{Tn}-measure as

$$dL(t,T_i-\delta,T_i) = \beta\left(t,T_i-\delta,T_i,L(t,T_j,\delta)\,|_{T_j\geq t}\right)dz(t,T_i)$$
$$= \beta\left(t,T_i-\delta,T_i,L(t,T_j,\delta)\,|_{T_j\geq t}\right)\left[dz(t,T_n)-\sum_{j=1}^{n-1}\frac{\delta\beta\left(t,T_j,T_{j+1},L(t,T_j,\delta)\,|_{T_j\geq t}\right)}{1+\delta L(t,T_j,T_{j+1})}dt\right]$$
$$= -\sum_{j=i}^{n-1}\frac{\delta\beta\left(t,T_i-\delta,T_i,L(t,T_k,\delta)\,|_{T_k\geq t}\right)\beta\left(t,T_j,T_{j+1},L(t,T_k,\delta)\,|_{T_k\geq t}\right)}{1+\delta L(t,T_j,T_{j+1})}dt$$
$$+\beta\left(t,T_i-\delta,T_i,L(t,T_k,\delta)\,|_{T_k\geq t}\right)dz(t,T_n)$$

Note that the drift may involve some or all of the other modelled forward rates. Therefore, the vector of all the forward rates ($L(t, T_0, T_1)$,…, $L(t, T_{n-1}, T_n)$) will follow an n-dimensional diffusion process so that a LIBOR market model can be represented as an n-factor diffusion model. Security prices are hence solutions to a partial differential equation (PDE), but in typical applications the dimension n, i.e. the number of forward rates, is so big that neither explicit nor numerical solution of the PDE is feasible. For example, in order to price caps, floors, and swaptions that depend on 3-month interest rates and have maturities of up to 10 years, one must model 40 forward rates so that the model is a 40-factor diffusion model! However, Andersen and Andreasen (2000) introduce a trick that may reduce the computational complexity considerably.

Next, let us consider an asset with a single payoff at some point in time $T \in [T_0, T_n]$. The payoff $N(T)$ may in general depend on the value of all the modelled forward rates at, and before time T. Let $V(t)$ denote the value of this asset at time t (measured in monetary units, e.g. dollars). From the definition of the T_n-forward martingale measure Q^{Tn} it follows that

$$V(t) = p(t,T_n)E^{Q^{T_n}}\left[\frac{N(T)}{p(T,T_n)}\,|\,\mathcal{F}_t\right]$$

In particular, if T is one of the time points of the tenor structure, say $T = T_k$, we get

$$V(t) = p(t,T_n)E^{Q^{T_n}}\left[\frac{N(T_k)}{p(T_k,T_n)}\,|\,\mathcal{F}_t\right]$$

From

$$L(t,T_i-\delta,T_i) = \frac{1}{\delta}\left(\frac{p(t,T_i-\delta)}{p(t,T_i)}-1\right)$$

we have that

$$\frac{1}{p(T_k,T_n)} = \frac{p(T_k,T_k)}{p(T_k,T_{k+1})}\frac{p(T_k,T_{k+1})}{p(T_k,T_{k+2})}\cdots\frac{p(T_k,T_{n-1})}{p(T_k,T_n)}$$

$$= \left[1+\delta L(T_k,T_k,T_{k+1})\right]\left[1+\delta L(T_k,T_{k+1},T_{k+2})\right]\ldots\left[1+\delta L(T_k,T_{n-1},T_n)\right]$$

$$= \prod_{j=k}^{n-1}\left[1+\delta L(T_k,T_j,T_{j+1})\right]$$

so that the price can be rewritten as

$$V(t) = p(t,T_n)E^{Q^{T_n}}\left[N(T_k)\prod_{j=k}^{n-1}\left[1+\delta L(T_k,T_j,T_{j+1})\right] \mid \mathcal{F}_t\right]$$

The right-hand side may be approximated using Monte Carlo simulations in which the evolution of the forward rates under Q^{Tn} is used, as outlined above. If the security matures at time T_n, the price expression is simpler

$$V(t) = p(t,T_n)E^{Q^{T_n}}\left[N(T_n) \mid \mathcal{F}_t\right]$$

In that case it suffices to simulate the evolution of the forward rates that determine the payoff of the security.

23.2.3.1 The spot LIBOR martingale measure

The spot martingale measure Q, which we defined above, is associated with the use of a bank account earning the continuously compounded short rate as the numeraire. However, the LIBOR market model does not at all involve the short rate so the traditional spot martingale measure does not make sense in this context. The LIBOR market counterpart is a roll over strategy in the shortest zero-coupon bonds. To be more precise, the strategy is initiated at time T_0 by an investment of one dollar in the zero-coupon bond maturing at time T_1, which allows for the purchase of $1/p(T_0, T_1)$ units of the bond. At time T_1 the payoff of $1/p(T_0, T_1)$ dollars is invested in the zero-coupon bond maturing at time T_2, etc. Let us define

$$I(t) = \min\{i \in \{1, 2, \ldots, n\} : T_i \geq t\}$$

so that $T_{I(t)}$ denotes the next payment date after time t. In particular, $I(T_i) = i$ so that $T_{I(T_i)} = T_i$. At any time $t \geq T_0$ the strategy consists of holding

$$M(t)=\frac{1}{p(T_0,T_1)}\frac{1}{p(T_1,T_2)}\cdots\frac{1}{p(T_{I(t)-1},T_{I(t)})}$$

units of the zero-coupon bond maturing at time $T_{I(t)}$. The value of this position is

$$\begin{aligned}A^*(t) &= p(t,T_{I(t)})M(t)=p(t,T_{I(t)})\prod_{j=0}^{I(t)-1}\frac{1}{p(T_j,T_{j+1})}\\ &= p(t,T_{I(t)})\prod_{j=0}^{I(t)-1}\left[1+\delta L(T_j,T_j,T_{j+1})\right]\end{aligned}$$

where the last equality follows from

$$L(t,T_i-\delta,T_i)=\frac{1}{\delta}\left(\frac{p(t,T_i-\delta)}{p(t,T_i)}-1\right)$$

Since $A^*(t)$ is positive, it is a valid numeraire. The corresponding martingale measure is called the **spot LIBOR martingale measure** and is denoted by Q^*.

Let us look at a security with a single payment at a time $T \in [T_0, T_n]$. The payoff $N(T)$ may depend on the values of all the modelled forward rates at, and before time T. Let us by $V(t)$ denote the dollar value of this asset at time t. From the definition of the spot LIBOR martingale measure Q^* it follows that

$$\frac{V(t)}{A^*(t)}=E^{Q^*}\left[\frac{N(T)}{A^*(T)}\mid \mathcal{F}_t\right]$$

and hence

$$V(t)=E^{Q^*}\left[\frac{A^*(t)}{A^*(T)}N(T)\mid \mathcal{F}_t\right]$$

From the calculation

$$\begin{aligned}\frac{A^*(t)}{A^*(T)} &= \frac{p(t,T_{I(t)})}{p(T,T_{I(T)})}\frac{\prod_{j=0}^{I(t)-1}\left[1+\delta L(T_j,T_j,T_{j+1})\right]}{\prod_{j=0}^{I(T)-1}\left[1+\delta L(T_j,T_j,T_{j+1})\right]}\\ &= \frac{p(t,T_{I(t)})}{p(T,T_{I(T)})}\prod_{j=I(t)}^{I(T)-1}\left[1+\delta L(T_j,T_j,T_{j+1})\right]^{-1}\end{aligned}$$

we get that the price can be rewritten as

$$V(t) = p(t,T_{I(t)})E^{Q^*}\left[\frac{N(T)}{p(T,T_{I(T)})}\prod_{j=I(t)}^{I(T)-1}\left[1+\delta L(T_j,T_j,T_{j+1})\right]^{-1} \mid \mathcal{F}_t\right]$$

In particular, if T is one of the dates in the tenor structure, say $T = T_k$, we get

$$V(t) = p(t,T_{I(t)})E^{Q^*}\left[N(T_k)\prod_{j=I(t)}^{I(T)-1}\left[1+\delta L(T_j,T_j,T_{j+1})\right]^{-1} \mid \mathcal{F}_t\right]$$

since $I(T_k) = k$ and $p(T_k,T_{I(T_k)}) = p(T_k,T_k) = 1$

In order to compute (typically by simulation) the expected value on the right-hand side, we need to know the evolution of the forward rates $L(t, T_j, T_{j+1})$ under the spot LIBOR martingale measure Q^*. It can be shown that the process z^* defined by

$$dz^*(t) = dz^{T_i}(t) - \left[\sigma^{T_{I(t)}}(t) - \sigma^{T_i}(t)\right]$$

is a standard Brownian motion under the probability measure Q^*. As usual, $\sigma^T(t)$ denotes the volatility of the zero-coupon bond maturing at time T. Repeated use of the theorem above yields

$$\sigma^{T_{I(t)}}(t) - \sigma^{T_i}(t) = \sum_{j=I(t)}^{i-1}\frac{\delta\beta\left(t,T_j,T_{j+1},L(t,T_k,\delta)\,|_{T_k\geq t}\right)}{1+\delta L(t,T_j,T_{j+1})}$$

so that

$$dz^*(t) = dz(t,T_i) - \sum_{j=I(t)}^{i-1}\frac{\delta\beta\left(t,T_j,T_{j+1},L(t,T_k,\delta)\,|_{T_k\geq t}\right)}{1+\delta L(t,T_j,T_{j+1})}dt$$

Substituting this relation into

$$dL(t,T_i-\delta,T_i) = \beta\left(t,T_i-\delta,T_i,L(t,T_j,\delta)\,|_{T_j\geq t}\right)dz(t,T_i), \quad i = 1\ldots,n.$$

we can rewrite the dynamics of the forward rates under the spot LIBOR martingale measure as

$$dL(t,T_i-\delta,T_i) = \beta\left(t,T_i-\delta,T_i,L(t,T_k,\delta)|_{T_k\geq t}\right)dz(t,T_i)$$

$$= \beta\left(t,T_i-\delta,T_i,L(t,T_k,\delta)|_{T_k\geq t}\right)\left[dz^*(t) - \sum_{j=I(t)}^{i-1}\frac{\delta\beta\left(t,T_j,T_{j+1},L(t,T_k,\delta)|_{T_k\geq t}\right)}{1+\delta L(t,T_j,T_{j+1})}dt\right]$$

$$= -\sum_{j=I(t)}^{i-1}\frac{\delta\beta\left(t,T_i-\delta,T_i,L(t,T_k,\delta)|_{T_k\geq t}\right)\beta\left(t,T_j,T_{j+1},L(t,T_k,\delta)|_{T_k\geq t}\right)}{1+\delta L(t,T_j,T_{j+1})}dt$$

$$+\beta\left(t,T_i-\delta,T_i,L(t,T_k,\delta)|_{T_k\geq t}\right)dz^*(t)$$

Note that the drift in the forward rates under the spot LIBOR martingale measure follows from the specification of the volatility function β and the current forward rates. The relation between the drift and the volatility is the market model counterpart to the drift restriction of the HJM models.

23.2.4 Consistent pricing

As indicated above, the model can be used for the pricing of all securities that only have payment dates in the set $\{T_1, T_2,..., T_n\}$, and where the size of the payment only depends on the modelled forward rates and no other random variables. This is true for caps and floors on δ-period interest rates of different maturities where the price can be computed as above. The model can also be used for the pricing of swaptions that expire on one of the dates T_0, T_1,..., T_{n-1}, and where the underlying swap has payment dates in the set $\{T_1,..., T_n\}$ and is based on the δ-period interest rate. For European swaptions the price can be written as

$$V(t) = p(t,T_{I(t)})E^{Q^*}\left[N(T_k)\prod_{j=I(t)}^{I(T)-1}\left[1+\delta L(T_j,T_j,T_{j+1})\right]^{-1} | \mathcal{F}_t\right]$$

For Bermuda swaptions that can be exercised at a subset of the swap payment dates $\{T_1,..., T_n\}$, one must maximize the right-hand side over all feasible exercise strategies. See Andersen (2000) for details and a description of a relatively simple Monte Carlo based method for the approximation of Bermuda swaption prices.

The LIBOR market model is built on assumptions about the forward rates over the time intervals $[T_0, T_1]$, $[T_1, T_2]$,..., $[T_{n-1}, T_n]$. However, these forward rates determine the forward rates for periods that are obtained by connecting succeeding intervals. For example, that the forward rate over the period $[T_0, T_2]$ is uniquely determined by the forward rates for the periods $[T_0, T_1]$ and $[T_1, T_2]$ since

$$L(t,T_0,T_2) = \frac{1}{T_2 - T_0}\left(\frac{p(t,T_0)}{p(t,T_2)} - 1\right) = \frac{1}{T_2 - T_0}\left(\frac{p(t,T_0)}{p(t,T_1)}\frac{p(t,T_1)}{p(t,T_2)} - 1\right)$$
$$= \frac{1}{2\delta}\left(\left[1 + \delta L(t,T_0,T_1)\right]\left[1 + \delta L(t,T_1,T_2)\right] - 1\right)$$

where $\delta = T_1 - T_0 = T_2 - T_1$ as usual. Therefore, the distributions of the forward rates $L(t, T_0, T_1)$ and $L(t, T_1, T_2)$ implied by the LIBOR market model determine the distribution of the forward rate $L(t, T_0, T_2)$. A LIBOR market model based on three-month interest rates can hence also be used for the pricing of contracts that depend on six-month interest rates, as long as the payment dates for these contracts are in the set $\{T_0, T_1, ..., T_n\}$. More generally, in the construction of a model, one is only allowed to make exogenous assumptions about the evolution of forward rates for non-overlapping periods.

23.3 The lognormal LIBOR market model

23.3.1 Black's model

The standard model for valuing OTC interest rate options, caps, floors and European swaptions, is the Black model. The Black model is used by traders in the market to price these derivatives and as will be seen later on, the analytical Black formulas will play a key role when calibrating the LIBOR Market model.

The basic assumptions under the Black model are:

- The underlying forward rate or swap rate is a log normally distributed stochastic variable.
- The volatility of the underlying is constant.
- Prices are arbitrage free.
- There is continuous trading in all instruments.

In Black's world we denote the forward/futures price with expiry T on an underlying with expiry T^* as $\Phi(T, T^*)$. The price is lognormally distributed with standard deviation $\sigma\sqrt{T-t}$. It is further assumed that on expiry, the expected futures price is equal to the current futures price

$$E^Q\left[\Phi(T,T^*) \mid \mathcal{F}_t\right] = \Phi(t,T^*)$$

For a European call option in Black's model we have

$$C_t = e^{-r(T-t)}\left\{\Phi(t,T^*)\cdot N\left(d_1\left[t,\Phi(t,T^*)\right]\right) - K \cdot N\left(d_2\left[t,\Phi(t,T^*)\right]\right)\right\}$$

where

$$d_2 = \frac{\ln(\Phi(t,T^*)/K) - (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

23.3.2 Bond Options

The unique no-arbitrage value at time t of a forward contract with delivery at time T of a zero coupon bond maturing at time S at delivery price K is given by

$$V(t,T,S) = p(t,S) - K \cdot p(t,T)$$

The unique no-arbitrage forward price on the zero coupon bond is

$$F(t,T,S) = \frac{p(t,S)}{p(t,T)}$$

Next, we consider a forward contract on a coupon bond where we assumed to yield payments at $T_1 < T_2 <...< T_n$ in time where $T < T_n$. We denote the coupon payments as c_i, $i = 1, 2,\ldots, n$. At time t the value of the bond is therefore given by

$$P(t) = \sum_{T_i > t} c_i p\left(t,T_i\right)$$

Let $V^{cpn}(t, T)$ denote the time t value of this forward contract. Then we have a no-arbitrage value if the forward given by

$$V^{cpn}(t,T) = \sum_{T_i > t} c_i p\left(t,T_i\right) - K \cdot p(t,T) = P(t) - \sum_{t<T_i<T} c_i p\left(t,T_i\right) - K \cdot p(t,T)$$

The no-arbitrage forward price is given by

$$F^{cpn}(t,T) = \frac{\sum_{T_i > T} c_i p\left(t,T_i\right)}{p\left(t,T\right)} = \frac{P(t) - \sum_{t<T_i<T} c_i p\left(t,T_i\right)}{p\left(t,T\right)} = \sum_{T_i > T} c_i F(t,T,T_i)$$

Consider between time t and delivery time T, the two portfolios

1. A forward contract, K zero coupon bonds maturing at T and for each T_i with $t < T_i < T$, c_i zero coupon bonds maturing at T_i
2. The underlying coupon bond.

These portfolios have exactly the same payments. At time T, the first portfolio equals $P(T) - K + K = P(T)$, which is identical to the value of the second portfolio. Therefore the absence of arbitrage implies

$$V^{cpn}(t,T) + K \cdot p(t,T) + \sum_{t<T_i<T} c_i \cdot p(t,T_i) = P(t)$$

The expected payoff of the forward contract is then given by

$$E^Q\left[\max\left(P(T)-K,0\right) \mid \mathcal{F}_t\right] = F^{cpn}(t,T)\cdot N\left(d_1\left[t,F^{cpn}(t,T)\right]\right) - K \cdot N\left(d_2\left[t,F^{cpn}(t,T)\right]\right)$$

where

$$d_2 = \frac{\ln(F^{cpn}(t,T)/K) - (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

If we multiply the expected payoff with the relevant discount factor, e.g. the zero coupon bond price $p(t, T)$ we get **Black's formula for a European call option on a bond**

$$\begin{aligned} C^{cpn}(t,T,K) &= p(t,T)\begin{Bmatrix} F^{cpn}(t,T)\cdot N\left(d_1\left[t,F^{cpn}(t,T)\right]\right) \\ -K\cdot N\left(d_2\left[t,F^{cpn}(t,T)\right]\right) \end{Bmatrix} \\ &= \left(P(t) - \sum_{t<T_i<T} c_i \cdot p(t,T_i)\right) N\left(d_1\left[t,F^{cpn}(t,T)\right]\right) \\ &\quad - K\cdot p(t,T) N\left(d_2\left[t,F^{cpn}(t,T)\right]\right) \end{aligned}$$

Similarly, for a put option we have

$$\begin{aligned} P^{cpn}(t,T,K) &= K\cdot p(t,T) N\left(-d_2\left[t,F^{cpn}(t,T)\right]\right) \\ &\quad - \left(P(t) - \sum_{t<T_i<T} c_i \cdot p(t,T_i)\right) N\left(-d_1\left[t,F^{cpn}(t,T)\right]\right) \end{aligned}$$

23.3.3 Caps and Floors

For a caplet, with a payoff given by

$$C^i_{T_i} = N\delta \max\left\{L(T_i, T_i - \delta) - K, 0\right\}$$

we obtain the Black's price as

$$C^i(t) = N\delta\, p(t,T_i)\begin{Bmatrix} L(t,T_i-\delta,T_i)\cdot N\left(d_1^i\left[t,L(t,T_i-\delta,T_i)\right]\right) \\ -K\cdot N\left(d_2^i\left[t,L(t,T_i-\delta,T_i)\right]\right) \end{Bmatrix}$$

where $t < T_i < \delta$ and

$$d_2^i = \frac{\ln(L(t,T_i-\delta,T_i)/K)-(\sigma_i^2/2)(T_i-\delta-t)}{\sigma\sqrt{T_i-\delta-t}} = d_1-\sigma_i\sqrt{T_i-\delta-t}$$

We have assumed that the forward rates $F(t, T_i - \delta, T_i)$ in a risk free world follow the process

$$dL(t,T_i-\delta,T_i) = \sigma_i L(t,T_i-\delta,T_i)dV(t)$$

The price of the cap and the floor is given by

$$C(t) = N\delta\sum_{i=1}^{n} p(t,T_i)\begin{Bmatrix} L(t,T_i-\delta,T_i)\cdot N\left(d_1^i\left[t,L(t,T_i-\delta,T_i)\right]\right) \\ -K\cdot N\left(d_2^i\left[t,L(t,T_i-\delta,T_i)\right]\right) \end{Bmatrix}$$

$$F(t) = N\delta\sum_{i=1}^{n} p(t,T_i)\begin{Bmatrix} K\cdot N\left(-d_2^i\left[t,L(t,T_i-\delta,T_i)\right]\right) \\ -L(t,T_i-\delta,T_i)\cdot N\left(-d_1^i\left[t,L(t,T_i-\delta,T_i)\right]\right) \end{Bmatrix}$$

23.3.4 LMM Model description

The traditional derivation of Black’s formula is based on some inappropriate assumptions.

First, the assumed log-normality of bond prices and interest rates is doubtful. For several reasons the price of a bond cannot follow a geometric Brownian motion throughout its life. We know that the price converges to the terminal payment of the bond as the maturity date approaches. Furthermore, the bond price is limited from above by the sum of the future bond payments under the appropriate assumption that all forward rates are non-negative. When the bond price approaches its upper limit or the maturity date approaches, the volatility of the bond price has to go to zero. The volatility of the bond price will therefore depend on both the level of the price and the time to maturity. The log-normality assumption can at most be an a locally relevant approximation to the true distribution. In addition, the forward price and the futures price on a bond are not necessarily equal when the interest rate uncertainty is taken into account. It is less clear whether it is reasonable to assume that future interest rates are log-normally distributed, and that the expected changes in the forward rates and the forward swap rates are zero in a risk-neutral world.

Second, the multiplication of the current discount factor and the risk-neutral expectation of the payoff do not lead to the correct price. In fact, this is true only if we take the expectation under the appropriate forward martingale measure instead of the risk-neutral measure.

Third, simultaneous applications of Black's formula to different derivative securities are mutually inconsistent. If, for example, we apply Black's formula to the pricing of a European option on zero-coupon bond, we must assume that the price of the zero-coupon bond is log-normally distributed. If we also apply Black's formula for the pricing of a European option on a coupon bond, we must assume that the price of the coupon bond is log-normally distributed. Since the price of the coupon bond is a weighted average of the prices of zero-coupon bonds and a sum of log-normally distributed random variables is not log-normally distributed, these assumptions are inconsistent. Similarly, the swap rate is a linear combination of forward rates. When Black's formula is applied for the pricing of caplets, it is implicitly assumed that the relevant forward rates are log-normally distributed. Then the swap rate will not be log-normally distributed, so that it is inconsistent to use Black's formula for swaptions also. Furthermore, log-normality assumptions for both interest rates and bond prices are inconsistent.

Several research papers suggest other models for bond option pricing that are also based on specific assumptions on the evolution of the price of the underlying bond. The most prominent examples are Ball and Torous (1983) and Schaefer and Schwartz (1987). A critical analysis of such models can be seen in Rady and Sandmann (1994). A problem in applying these models is that the assumptions about the price dynamics for different bonds may be inconsistent, and hence the option pricing formula obtained in the model will only be valid for options on one particular bond.

To ensure consistent pricing of different fixed income securities we must model the evolution of the entire term structure of interest rates. In many of the consistent term structure models we shall discuss in the following sections, we will obtain relatively simple and internally consistent pricing formulas for many of the popular fixed income securities. As we shall see in this section, it is in fact possible to construct consistent term structure models in which Black's formula is the correct pricing formula for some securities, but, even in those models, applications of Black's formula for different classes of securities are inconsistent.

The log-normal LIBOR market model provides a more reasonable framework in which the Black cap formula is valid. The model was originally developed by Miltersen, Sandmann, and Sondermann (1997), while Brace, Gatarek, and Musiela (1997) sorted out some technical details and introduced an explicit, but approximative, expression for the prices of European swaptions in the lognormal LIBOR market model. Whereas Miltersen, Sandmann, and Sondermann derive the cap price formula using PDEs, we will follow the approach taken by Brace, Gatarek, and Musiela and use the forward martingale measure technique, since this simplifies the analysis considerably.

In the development of Black's cap pricing formula, we assumed among other things that the forward rate $L(t, T_i - \delta, T_i)$ was a martingale under the spot martingale measure Q and that the future value $L(T_i - \delta, T_i - \delta, T_i)$ was log-normally distributed under Q. However, as was seen in the theorem above this forward rate is a martingale under the T_i-forward martingale measure Q^{Ti} and will therefore not be a martingale under the Q-measure. (Remember: an equivalent change of measure corresponds to changing the drift rate.) Looking at the general cap pricing formula

$$C(t) = N\delta \sum_{i=1}^{n} p(t,T_i) E_t^{Q^{T_i}} \left[\max\left\{ L(T_i - \delta, T_i - \delta, T_i) - K, 0 \right\} \right]; \qquad t < T_0$$

it is clear that we can obtain a pricing formula of the same form as Black's formula by assuming that $L(T_i - \delta, T_i - \delta, T_i)$ is lognormally distributed under the T_i-forward martingale measure Q^{Ti} . This is exactly the assumption of the lognormal LIBOR market model

$$dL(t, T_i - \delta, T_i) = L(t, T_i - \delta, T_i)\gamma\left(t, T_i - \delta, T_i\right) dz(t, T_i), \quad i = 1, 2..., n.$$

where $\gamma(t, T_i\text{-}\delta, T_i)$ is a bounded, deterministic function. Here we assume that the relevant forward rates are only affected by one Brownian motion, but below we shall briefly consider multi-factor lognormal LIBOR market models.

A familiar application of Itô's lemma implies that

$$d\left[\ln L(t, T_i - \delta, T_i)\right] = -\frac{1}{2}\left(\gamma\left(t, T_i - \delta, T_i\right)\right)^2 dt + \gamma\left(t, T_i - \delta, T_i\right) dz(t, T_i)$$

from which we see that

$$\ln L(T_i - \delta, T_i - \delta, T_i) = \ln L(t, T_i - \delta, T_i) - \frac{1}{2}\int_t^{T_i - \delta} \left(\gamma\left(u, T_i - \delta, T_i\right)\right)^2 du + \int_t^{T_i - \delta} \gamma\left(u, T_i - \delta, T_i\right) dz(u, T_i)$$

Because γ is a deterministic function, it follows that

$$\int_t^{T_i - \delta} \gamma\left(u, T_i - \delta, T_i\right) dz(u, T_i) \sim N\left[0, \int_t^{T_i - \delta} \left(\gamma\left(u, T_i - \delta, T_i\right)\right)^2 du\right]$$

under the T_i-forward martingale measure. Hence,

$$\ln L(T_i-\delta,T_i-\delta,T_i) = N\begin{bmatrix} \ln L(t,T_i-\delta,T_i)-\frac{1}{2}\int\limits_t^{T_i-\delta}\left(\gamma\left(u,T_i-\delta,T_i\right)\right)^2 du, \\ \int\limits_t^{T_i-\delta}\left(\gamma\left(u,T_i-\delta,T_i\right)\right)^2 du \end{bmatrix}$$

so that T_i - δ is log-normally distributed under Q^{Ti}. The following result should not now come as a surprise

Theorem 23.3. *Under the assumption*

$$dL(t,T_i-\delta,T_i) = L(t,T_i-\delta,T_i)\gamma\left(t,T_i-\delta,T_i\right)dz(t,T_i), \quad i=1,2...,n.$$

the price of the caplet with payment date T_i at any time $t < T_i - \delta$ is given by

$$C^i(t) = N\delta p(t,T_i)\left\{L(t,T_i-\delta,T_i)\cdot N\left(d_1^i\right)-K\cdot N\left(d_2^i\right)\right\}$$

where

$$\begin{cases} d_1^i = \dfrac{\ln(L(t,T_i-\delta,T_i)/K)}{v_L(t,T_i-\delta,T_i)}+\dfrac{1}{2}v_L(t,T_i-\delta,T_i) \\ d_2^i = d_1^i - v_L(t,T_i-\delta,T_i) \end{cases}$$

and

$$v_L(t,T_i-\delta,T_i) = \sqrt{\int\limits_t^{T_i-\delta}\left(\gamma\left(u,T_i-\delta,T_i\right)\right)^2 du}$$

Note that $v_L(t,\ T_i - \delta,\ T_i)^2$ is the variance of $\ln[L(T_i - \delta,\ T_i - \delta,\ T_i)]$ under the T^i-forward martingale measure given the information available at time t. The caplet price above is identical to Black's formula if we insert

$$\sigma_i = \frac{v_L(t,T_i-\delta,T_i)}{\sqrt{T_i-\delta-t}}$$

An immediate consequence of the theorem above is the following cap pricing formula in the log-normal one-factor LIBOR market model

Theorem 23.4. *Under the assumption*

$$dL(t,T_i-\delta,T_i) = L(t,T_i-\delta,T_i)\gamma\left(t,T_i-\delta,T_i\right)dz(t,T_i), \quad i=1,2...,n.$$

the price of a cap at any time $t < T_0$ is given as

$$C(t) = N\delta \sum_{I=1}^{N} p(t,T_i)\left\{L(t,T_i-\delta,T_i)\cdot N\left(d_1^i\right) - K\cdot N\left(d_2^i\right)\right\}$$

where d_1^i and d_2^i are as above.

For $t \geq T_0$ the first upcoming payment of the cap is known and is therefore to be discounted with the relevant discount factor, while the remaining payments are to be valued as above.

Analogously, the price of a floor is

$$F(t) = N\delta \sum_{I=1}^{N} p(t,T_i)\left\{K\cdot N\left(-d_2^i\right) - L(t,T_i-\delta,T_i)\cdot N\left(-d_1^i\right)\right\}$$

The deterministic function $\gamma(t,\ T_i - \delta,\ T_i)$ remains to be specified. We will discuss this matter below.

If the term structure is affected by n exogenous standard Brownian motions, the assumption on $dL(t,\ T_i - \delta,\ T_i)$ above is replaced by

$$dL(t,T_i-\delta,T_i) = L(t,T_i-\delta,T_i)\sum_{j=1}^{n}\gamma_j\left(t,T_i-\delta,T_i\right)dz_j(t,T_i)$$

where all $\gamma_j(t,\ T_i - \delta,\ T_i)$ are bounded and deterministic functions. Again, the cap price is given by the formula above with the small change that $v_L(t,\ T_i - \delta,\ T_i)$ is to be computed as

$$v_L(t,T_i-\delta,T_i) = \sqrt{\sum_{j=1}^{n}\int_{t}^{T_i-\delta}\left(\gamma_j\left(u,T_i-\delta,T_i\right)\right)^2 du}$$

23.3.5 Pricing of other securities

No exact explicit solution for European swaptions has been found in the lognormal LIBOR market setting. In particular, Black's formula for swaptions is not correct under the assumption on $dL(t,\ T_i - \delta,\ T_i)$. The reason is that when the forward LIBOR rates have volatilities proportional to their level, the volatility of the forward swap rate will not be proportional to the level of the forward swap rate. The swaption price can be approximated by a Monte Carlo simulation, which is often quite time-consuming. Brace, Gatarek, and Musiela (1997) derived the following Black-type approximation to the price of a European payer swaption with expiration date T_0 and exercise rate K under the lognormal LIBOR market model assumptions

$$P(t) = N\delta \sum_{I=1}^{N} p(t,T_i)\left\{L(t,T_i-\delta,T_i)\cdot N\left(d_1^i\right) + K\cdot N\left(d_2^i\right)\right\} \quad t < T_0$$

where d_1 and d_2 are quite complicated expressions involving the variances and covariance of the time T_0 values of the forward rates involved. These variances and covariance are determined by the γ-function. This approximation delivers the price much faster than a Monte Carlo simulation. Brace, Gatarek, and Musiela provide numerical examples in which the price computed using the approximation above is very close to the correct price (computed using Monte Carlo simulations). Of course, a similar approximation also applies to the European receiver swaption.

Under the assumptions of the lognormal LIBOR market model Miltersen, Sandmann, and Sondermann (1997) derived an explicit pricing formula for European options on zero-coupon bonds, but only for options expiring at one of the time points T_0, T_1, ... , T_{n-1}, and where the underlying zero-coupon bond matures at the following date in this sequence. In other words, the time distance between the maturity of the option and the maturity of the underlying zero-coupon bond must be equal to δ. Representing the exercise price by K, the pricing formula for a European call option is

$$C^i(t,K,T_i-\delta,T_i) = \left(1-K\right)p(t,T_i)N\left(e_1^i\right) - K\cdot\left[p(t,T_i-\delta)-p(t,T_i)\right]N\left(e_2^i\right)$$

where

$$\begin{cases} e_1^i = \dfrac{1}{v_L(t,T_i-\delta,T_i)}\ln\left(\dfrac{(1-K)p(t,T_i)}{K\cdot\left[p(t,T_i-\delta)-p(t,T_i)\right]}\right) + \dfrac{1}{2}v_L(t,T_i-\delta,T_i) \\ e_2^i = e_1^i - v_L(t,T_i-\delta,T_i) \end{cases}$$

and

$$v_L(t,T_i-\delta,T_i) = \sqrt{\int_t^{T_i-\delta}\left(\gamma\left(u,T_i-\delta,T_i\right)\right)^2 du}$$

or

$$v_L(t,T_i-\delta,T_i) = \sqrt{\sum_{j=1}^{n}\int_t^{T_i-\delta}\left(\gamma_j\left(u,T_i-\delta,T_i\right)\right)^2 du}$$

The price of the corresponding European put option follows from the put-call parity.

23.3.6 Further remarks

De Jong, Driessen, and Pelsser (2001) investigated the extent to which different lognormal LIBOR and swap market models can explain empirical data consisting of forward LIBOR interest rates, forward swap rates, and prices of caplets and European swaptions. The observations are from the U.S. market in 1995 and 1996. For the lognormal one-factor LIBOR market model they find that it is empirically more appropriate to use a γ-function that is exponentially decreasing in the time-to-maturity $T_i - \delta - t$ of the forward rates,

$$\gamma(t, T_i - \delta, T_i) = \gamma e^{-\kappa(T_i - \delta - t)},$$

than to use a constant, $\gamma(t, T_i - \delta, T_i) = \gamma$. This is related to the well-documented mean reversion of interest rates that makes long term interest rates relatively less volatile than shorter term interest rates. They also calibrate two similar model specifications perfectly to observed caplet prices, but find that in general the prices of swaptions in these models are further from the market prices than are the prices in the time homogeneous models above. In all cases the swaption prices computed using one of these lognormal LIBOR market models exceed the market prices, i.e. the lognormal LIBOR market models overestimate the swaption prices. All their specifications of the lognormal one-factor LIBOR market model give a relatively inaccurate description of market data and are rejected by statistical tests. De Jong, Driessen, and Pelsser also show that two-factor lognormal LIBOR market models are not significantly better than the one-factor models and conclude that the lognormality assumption is probably inappropriate. Finally, they present similar results for lognormal swap market models and find that these models are even worse than the lognormal LIBOR market models when it comes to fitting the data.

23.4 Calibrating the LIBOR Market Model

In this section we describe how to calibrate LMM to market data. A basic assumption is that we have chosen the Forward-Rate-Based Libor Market Model. This is the natural approach when pricing caps and floors both not when pricing swaptions.

Let the tenor structure be $0 = T_0 < T_1 < \ldots < T_{n-1} < T_n$ and i an integer ranging over the reset dates of the rates, e.g.$1 \leq i \leq n$.

We define $\eta(t)$ as the unique index such that $T_{\eta(t)}$ is the next tenor date after t. The (one factor) model is given by the following stochastic differential equation (SDE) for the underlying rates (swap or forward)

$$\frac{df_i}{f_i} = \mu_i\left(f(t), t\right)dt + \sigma_i(t)dz(t)$$

where

f_i	=	forward/swap rate at time i
μ_i	=	drift term
σ_i	=	volatility of rate i

$z(t)$ = is a Wiener process

The solution to this stochastic differential equation (SDE) is

$$f_i(T) = f_i(T)\exp\left(\int_0^T \left(\mu_i(u) - \frac{1}{2}s_i^2(u)\right)du + \int_0^T s_i(u)\,dz(u)\right)$$

The drift terms depend on the choice of numeraire and can be determined by applying the assumption of no arbitrage. Suppose we have forward rates as the underlying rates and choose $p(T_0, T_1)$ as the numeraire. Then the drift terms become

$$\mu_i = \sigma_i \sum_{k=1}^{i} \frac{\sigma_k f_k (T_{k+1} - T_k)\rho_{ik}}{1 + f_k(T_{k+1} - T_k)}$$

Determining the time dependent forward rate volatilities is equivalent to calibrating the model. How the calibration is performed is explained in a section below.

Although it is not necessary it will always be assumed here that the (instantaneous) volatilities σ_i of the rates are deterministic (not stochastic) functions of time.

A one-factor model means that all the forward rates are perfectly instantaneously correlated. In this case, a single Wiener process is sufficient to evolve the rates. This is not often a reasonable assumption, and eliminates one of the advantages of employing the LIBOR Market model. A m-factor model is one where m independent Wiener processes are used to evolve the rates. In this case the equation becomes

$$\frac{df_i}{f_i} = \mu_i\left(f(t), t\right)dt + \sum_{k=1}^{m} \sigma_{i,k}(t)\,dz_k(t), \qquad 1 \le k \le m$$

This is solved for

$$f_i(T) = f_i(T)\exp\left(\int_0^T \left[\mu_i(u) - \frac{1}{2}\sigma_i^2(u)\right]du + \int_0^T \sum_{k=1}^{m} \sigma_{i,k}(u)\,dz_k(u)\right)$$

The loadings $\sigma_{i.k}(u)$, can be interpreted as the sensitivities at time u of the ith forward rate to the kth shock provided by the Wiener process z_k. They must satisfy

$$\sigma_i^2(t) = \sum_{k=1}^{m} \sigma_{i,k}^2(t)$$

All that remains before we can start to analyse how the rates will evolve is to specify the instantaneous volatilities σ_i's and their loadings $\sigma_{i,k}$'s. This can be done in many different ways. One choice is presented below.

23.4.1 Volatility Calibration

Volatility calibration deals with the determination of the σ_i's (the instantaneous volatility of the forward rate with reset at T_i). This is done differently for caps and swaptions. Since the cap volatility calibration is a first step in the swaption volatility calibration, we start by describing the cap volatility calibration.

23.4.2 Cap Volatility Calibration

In the Black model for caplets it is assumed that the underlying rate has a lognormal distribution with variance equal to $T \cdot \sigma_{Black}^2$ where T is the reset date of the underlying forward rate. In the LIBOR Market model, this lognormal assumption is also made for each rate separately. The instantaneous volatility at reset for each rate is related to the above expression in the following way

$$\int_0^{T_i} \sigma_i^2(t)dt = T_i \sigma_{Black}^2$$

In other words, the instantaneous volatility at reset for each underlying rate is equal to the implied Black volatility, which can be read from the market. Although not necessary, we make the assumption that the σ_i's are deterministic functions of time only. There are (infinitely) many solutions to these equations, and our goal is to pick one that fits our needs. We follow the approach suggested by Rebonato (2002). Let

$$\sigma(t) = \left(a + bt\right)e^{-ct} + d$$

and

$$\sigma_i\left(t\right) = k_i \sigma\left(T_i - t\right)$$

This form is flexible and can by varying the constants *a, b, c* and *d* take many different shapes. The constants k_i are rate specific and are used to assure that the caplet prices are exactly recovered. How the k_i's are set should become clear below.

1. Find values on the constants *a, b, c* and *d* such that the forward rates

$$\frac{df_i}{f_i} = \mu_i\left(f(t),t\right)dt + \sigma_i(t)dz(t)$$

fit as close as possible. We use both the Broyden-Fletcher-Goldfarb-Shannon dimensional variable metric method and the Levenberg-Marquardt method[44] in parallel to solve this problem and pick the best solution.

2. Set values on the k_i's by computing

$$k_i = \sqrt{\frac{\sigma^2_{Black} T_i}{\int_0^{T_i} \sigma_i^2(t)\,dt}}$$

The second step ensures equality for the forward rates, that is, the instantaneous volatility and the implied Black volatility are set to be equal at each reset. This completes the volatility calibration for caps.

Before we end the cap calibration section, we shall discuss the issue of deciding the implied Black volatility, e.g. σ_{Black}, in the case when we are calibrating to an exotic cap with path-dependent strikes. For example, consider a ratchet cap where each caplet has a strike given by $K_i = f_{i-1} + X$ where K_i and X denote the strike for the i:th caplet and a spread respectively. Recall that σ_{Black} of a caplet is a function of the maturity of the caplet and the strike.

The fact that σ_{Black} depends on the strike gives us some problems. To see this, note that in order to get σ_{Black} of a caplet we must know its strike. But if the strike is path-dependent, as it is in a ratchet cap, we cannot know the strike beforehand. To solve this problem the following approach is taken

1. Make good guesses on the start values of the strikes
2. Get the σ_{Black} of the caplets by using the strikes from the first step
3. Evolve a small sample of the rates, e.g. 1024 Monte-Carlo simulations. Then compute the average rates for each caplet.
4. Compute new strikes by using the average rates.
5. Go to the second step with the newly computed strikes and repeat until some desirable convergence criterion is achieved.

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[44] The Broyden-Fletcher-Goldfarb-Shannon dimensional variable metric sometimes gives adjusting factors with the property that the first ones (i.e. i = 1, 2, …) have a higher deviation from unity than the rest. The Levenberg-Marquardt method yields adjusting factors with a more constant deviation from unity.

Empirical results show that this scheme always (although not proven) converges and gives good estimates on the strikes. Let us now turn to the volatility calibration for swaptions.

23.4.3 Swaption Volatility Calibration

We concentrate on the volatility calibration of a Bermudan swaption. The volatility calibration of a European swaption is a special case of this discussion. The basic set-up of the calibration is that we want to recover the implied Black volatilities of a set of co-terminal swap rates. Let T_1, T_2,…,T_n be the expiry dates of the co-terminal swaptions and SR_i denote the i:th swap rate, e.g. the swap rate for the swaption with expiry T_i. Recall that

$$SR_i = \sum_{j=i}^{n} w_j f_j(t)$$

where

$$w_j = \frac{B_{j+1}(T_{j+1} - T_j)}{\sum_{k=i}^{n} B_{k+1}(T_{k+1} - T_k)}$$

Furthermore, denote with $\sigma_{i \times n}(t)$ the instantaneous volatility of SR_i, e.g. the instantaneous volatility of the swap rate with expiry at T_i and maturity at T_n. It can be shown that:

$$\sigma_{i \times n}(t)^2 = \sum_{j=i}^{n}\sum_{k=i}^{n} \zeta_{jk}(t)\rho_{jk}\sigma_j(t)\sigma_k(t)$$

where

$$\zeta_{jk}(t) = \frac{w_k(t)f_k(t)w_j(t)f_j(t)}{\left(\sum_{m=i}^{n} w_m(t)f_m(t)\right)^2}$$

The last equation is an approximation. One of the main results is that:

$$\sigma_{i \times n}(t)^2 \approx \sum_{j=i}^{n}\sum_{k=i}^{n} \zeta_{jk}(T_0)\rho_{jk}\sigma_j(t)\sigma_k(t)$$

This is the key point in the calibration and one ought to understand how this greatly simplifies our task (it is recommended to read Rebonato (2002) "Modern Pricing of Interest-Rate Derivatives", Princeton University Press for a discussion regarding this issue). In order to recover the swaption prices, we must have that

$$\sigma_{i\times n}^{Black}(t)^2 T_i = \int_0^{T_i} \sigma_{i\times n}(u)^2 du$$

Our approach to achieving this is the following

Find values of the constants *a, b, c* and *d* such that the equations for the forward rates fit as close as possible. We use both the Broyden-Fletcher-Goldfarb-Shannon dimensional variable metric method and the Levenberg-Marquardt method (for a description of these methods, refer to Press et. al, (2002) "Numerical Recipes in C") in parallel to solve this problem and pick the best solution. Note that this is the same first step as in the cap calibration.

Consider the last swaption in the set of co-terminal swaptions, e.g. with expiry T_n. This is simply a floating rate exchanged for a fixed rate, i.e. a caplet. Set

$$k_n = \sqrt{\frac{\sigma_{n\times n}^{Black}(t)^2 T_n}{\int_0^{T_n} \sigma_{n\times n}(u)^2 du}}$$

which makes

$$\sigma_i(t) = k_i \sigma(T_i - t)$$

for $i = n$ an equality.

Move a step back and consider the swaption with expiry T_{n-1}. Our goal is to set the value for k_{n-1} such that

$$\sigma_{n-1\times n}^{Black}(t)^2 T_{n-1} = \int_0^{T_{n-1}} \sigma_{n-1\times n}(u)^2 du .$$

Since we have already solved for k_n and are using the approximation of

$$\int_0^{T_i} \sigma_i^2(t)dt = T_i\sigma_{Black}^2 ,$$

k_{n-1} is the only unknown variable. Since k_{n-1} appears in squared form, we need to solve a second-degree equation. Although straightforward, the algebra becomes quite messy.

Next we continue to compute the values of the remaining k_i's in a similar fashion as in the previous step. Doing this yields values for the k_i's such that

$$\sigma_i(t) = k_i \sigma(T_i - t)$$

is fulfilled for $1 \le i \le n$.

Our approach to calibration of the instantaneous swap volatility cannot, as far as we know, be found in any published/academic/scientific article. This is a brief motivation why it seems to be a desirable procedure. In all but very rare cases, the first step gives values on *a, b, c* and *d* such that the instantaneous volatilities of the forward rates are time homogeneous. A problem might be that the constants *a, b, c* and *d* are not designed to recover the swaption prices. However, they should not be too far off since we are considering the same rates, namely the forward rates. Finally, the remaining steps make sure that we recover the swaption prices exactly.

23.4.4 Principal Component Analysis

We use principal component analysis to reduce the number of driving factors needed when valuing plain-vanilla caps, European swaptions, and Bermudan swaptions. In this section we describe how this is done. Note that we do not use principal component analysis for the valuation of path-dependent caps such as ratchet, sticky, momentum, flexi, and chooser. In this case, we use as many factors as there are rates. More information on why we do this can be found in Rebonato (2002) "Modern Pricing of Interest-Rate Derivatives", Princeton University Press.

Consider a cap with n caplets with resets at $T_1, T_2, \ldots, T_n$ respectively. Each caplet has an associated forward rate f_i for $1 \le i \le n$. We describe the principal component analysis in a simple case, namely when the rates $f_1, f_2, \ldots, f_n$ are evolved to the first reset date T_1. The complete picture can then be understood from this discussion. Now suppose we have an m-factor model where $m < n$ and let

$$\mathbf{Cov}_{ij} = \int_0^{T_1} \rho_{ij}\sigma_i(u)\sigma_j(u)du$$

be the $n \times n$ terminal covariance matrix where $1 \le i \le n$ and $1 \le j \le n$. Note that $\mathbf{Cov}_{ij}$ is symmetric.

Use the Jacobian Transformations of a Symmetric Matrix method to find the n eigenvalues of $\mathbf{Cov}_{ij}$ and the corresponding normalized eigenvectors as described by Rebonato (2002, section 11.1). Denote the vector of eigenvalues with $\mathbf{e}_i = [e_1 \; e_2 \; \ldots \; e_n]^T$ and the corresponding normalized eigenvectors with $\mathbf{v}_i$. Furthermore, let $\mathbf{V}_{ij} = [\mathbf{v}_1 \; \mathbf{v}_2 \; \ldots \; \mathbf{v}_n]$. Sort $\mathbf{e}_i$ in decreasing order and change the vectors in $\mathbf{V}_{ij}$ accordingly. Compute

$$\mathbf{B}_{ik} = \left[\sqrt{e_1}\mathbf{v}_1 \; \sqrt{e_2}\mathbf{v}_2 \; \cdots \; \sqrt{e_m}\mathbf{v}_m \right]$$

where $1 \le k \le m$. Compute

$$\mathbf{s}_i = \left[\sqrt{\frac{\mathbf{Cov}_{11}}{\sum_{l=1}^{m} e_l v_{1l}^2}} \; \sqrt{\frac{\mathbf{Cov}_{22}}{\sum_{l=1}^{m} e_l v_{2l}^2}} \; \ldots \; \sqrt{\frac{\mathbf{Cov}_{nn}}{\sum_{l=1}^{m} e_l v_{nl}^2}} \right]^T = \left[s_1 \; s_2 \; \ldots \; s_n \right]^T .$$

and

$$\mathrm{B}_{ik}^{'} = \begin{bmatrix} s_1\sqrt{e_1}v_{11} & s_1\sqrt{e_2}v_{12} & \ldots & s_1\sqrt{e_m}v_{1m} \\ s_2\sqrt{e_1}v_{21} & s_2\sqrt{e_2}v_{22} & \ldots & s_2\sqrt{e_m}v_{2m} \\ \vdots & & & \\ s_n\sqrt{e_1}v_{n1} & s_n\sqrt{e_2}v_{n2} & \ldots & s_n\sqrt{e_m}v_{nm} \end{bmatrix}.$$

Finally, compute the model covariance $\mathbf{Cov}_{ij}^{'} = \mathbf{B}_{ik}^{'} \times \mathbf{B}_{ik}^{'\,T}$. In particular note that $\mathbf{Cov}_{ii}^{'} = \mathbf{Cov}_{ii}$ for all *i*'s, that is, the variance of each rate is unchanged, and that $\mathbf{B}_{ik}^{'}$ is the principal component matrix. To put this in the context of the section 'Libor Market Model', the equation for the drift terms

$$\mu_i = \sigma_i \sum_{k=1}^{i} \frac{\sigma_k f_k (T_{k+1} - T_k) \rho_{ik}}{1 + f_k (T_{k+1} - T_k)}$$

is re-written as

$$\mu_i = \sum_{l=1}^{i} \frac{f_l (T_{l+1} - T_l)}{1 + f_l (T_{l+1} - T_l)} \mathbf{Cov}_{il}^{'}$$

and in equation

$$\frac{df_i}{f_i} = \mu_i \left(f(t), t \right) dt + \sum_{k=1}^{m} \sigma_{i,k}(t) dz_k(t), \qquad 1 \le k \le m$$

$\sigma_{i,k}$ correspond to $\mathbf{B}_{ik}^{'}$.

23.4.5 Correlation Matrix

The $n \times n$ correlation matrix

$$\mathbf{P}_{ij} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \\ \vdots & & \ddots & \\ \rho_{n1} & \cdots & & \rho_{nn} \end{bmatrix}$$

is user defined. After the user has input $\mathbf{P}_{ij}$, we use the heuristic of applying principal component analysis to it as described in the previous section before the calculations start. In other words, $\mathbf{P}_{ij}$ is exposed to the same transformation as $\mathbf{Cov}_{ij}$ was above.

23.5 Evolving the Forward Rates

We use two different approaches to evolve the forward rates – the short step and the long step. The short step approach evolves the 'living' forward rates to each reset. To exemplify, suppose that the resets of the rates are T_1, T_2,…,T_n. Then f_1, f_2,…,f_n are evolved to T_1,f_2,…,f_n are evolved to T_2, and so on. We do not attempt to describe the technical details of the approach here. However, the implications of the results are that we can evolve a forward rate to an appropriate point in time in one step, without hardly any loss of accuracy when compared to the short step approach.

We use both approaches when evolving the forward rates. In particular, we use the long step when valuing a cap (of any kind). This allows us to evolve each forward rate to its reset date in one step. When valuing a swaption (of any kind) we use the short step for all but the first time sensitive date (i.e. the first exercise date in the case of a Bermudan swaption); to the first time sensitive date when we use the long step.

23.6 Pricing of Bermudan Swaptions

In this section we make a theoretical elaboration of the pricing procedure for a Bermudan swaption. This pricing procedure is the only one we need to describe; a European swaption is, as we will see, a special case of this discussion, and the pricing procedure of a cap follows from its definition.

A Bermudan Swaption contract denoted by *X*-non-call-*Y* gives the holder the right to enter into a Swap at a pre-specified strike rate '*K*' on a number of exercise opportunities. The first exercise opportunity in this case would be *Y* years after inception. The swap that can be entered into always has the same terminal maturity date, X. A Bermudan Swaption entitling the holder the right to enter into a swap in which they pay the fixed rate is referred to as a *Payer's* otherwise *Receiver's*.

Therefore, as the owner of a Bermudan Swaption one is in a position where at each exercise date it is necessary to use one's own judgment to determine if exercise is optimal or not. This makes the pricing of Bermudan Swaptions a little more complicated than that of a European Swaption. We have implemented Andersen's strategy that can be found in Andersen (1999) A simple approach to the pricing of Bermudan swaptions in the multi-factor LIBOR Market Model", The Journal of Computational Finance, 3(2):5-32, 1999/2000. Below we describe the approach and how we use it.

When pricing a Bermudan Swaptions the most important question is how to determine the free exercise boundary. In other words, given that the world is in a certain state at one of the exercise dates, under what circumstances should one exercise the option?

Let:

$S_{s,e}$ = The European payer's Swaption maturing at time T_s and with a last cash flow at date T_e.

$S_{s,x,e}$ = The Bermudan Swaption with lockout date (first exercise oppor tunity) T_s, last exercise date T_x and final swap maturity T_e.

The decision whether or not to exercise a Bermudan Swaption at a date T_i will in general depend on the state of all forward rates $F_I(T_i)$. To simplify matters somewhat one could make the assumption that the strategy depends only on the intrinsic values of the underlying swap. Let $I(T_i)$, be the indicator function that equals one if exercise is optimal at dates T_i and zero otherwise. It is hence assumed that

$$I(T_i) = f\left(S_{i,e}, H(T_i)\right)$$

Where f is a specified Boolean function with a possibly time-dependent parameter H. The relationship:

$$I(T_x) = \begin{cases} 1 & S(T_x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

must of course be fulfilled. For the other exercise opportunities the following form of $I(T_i)$ is assumed:

$$I(T_i) = \begin{cases} 1 & S_{i,e}(T_i) > H(T_i) \\ 0 & \text{otherwise} \end{cases}$$

In this strategy the option is exercised if the intrinsic value of the underlying swap is above the barrier H.

The next step is to determine the value of the function H at the dates $T_s,\ldots,T_x$. $H(T)$. The function H is characterized as being the function that maximizes the value of the Bermudan Swaption.

A brute-force way of determining the values of the function H is hence to solve the multi-dimensional optimization problem. This could be done as follows. Store a Monte Carlo simulation in memory. Simultaneously find values on the function H at the dates $T_s,\ldots,T_x$ such that value of the Bermudan Swaption is maximized. This would be an iterative process where in each iteration, first, new values on H at the dates $T_s,\ldots,T_x$ are set, then, the value of the Bermudan Swaption is calculated. Whichever choices on H at the dates $T_s,\ldots,T_x$ that give the highest value on the Bermudan Swaption is picked. Obviously, this way of finding H is tremendously slow and not applicable in practice. Fortunately, there is an another way of finding H, as proposed by Andersen (1999), which is much more efficient. It can be described as follows:

1. Set $\boldsymbol{H}(T_x) = 0$.
2. Compute an appropriate Monte Carlo simulation of the forward rates and store it in memory.
3. Consider the Bermudan Swaption $S_{n-1,x,e}$. At time T_{n-1} the exercise strategy must be the same as for $S_{s,x,e}$ since at this time there is no difference between the options. $\boldsymbol{H}(T_x) = 0$ (Ordinary European option) is known and determining the value of $\boldsymbol{H}(T_{n-1})$ is hence a one-dimensional optimization problem. We solve this optimization problem with the Golden Section Search in One Dimension (see section 10.1 of Press et. al, (2002) "Numerical Recipes in C" for a description of this algorithm).
4. Repeat in turn the previous step for $S_{x-2,x,e}, S_{x-3,x,e}, \ldots, S_{s+1,x,e}, S_{s,x,e}$.

When doing this, it is sufficient to store a single Monte Carlo session in memory and to re-use it over and over again. Having determined the exercise boundary in this way another Monte Carlo simulation is run to calculate the price. This Monte Carlo approach to determine the free exercise boundary produces a lower bound on Bermudan Swaption prices that can be shown to be very tight for many realistic term structures.

Finally, note that valuing a European Swaption is a special case of valuing a Bermudan Swaption – a European Swaption is a Bermudan Swaption with one exercise event, T_x, and it follows that $\boldsymbol{H}(T_x) = 0$.