Chapter 19

The Black Model

19 Pricing Interest Rate Options using Black

The Black-76 modified Black-Scholes model has become the standard model for valuing OTC interest rate options, Caps, Floors and European Swaptions. The formula was originally developed to price options on forwards and assumes that the underlying asset is lognormal distributed.

Black's formula is often recalled as a special case of the Black and Scholes one, but it is in reality a generalization: if one applies Black to an equity option, where $S(T) = S(0)e^{rT}$ one gets the Black and Scholes formula. But the Black formula also holds when the spot has a complex dynamics and there is no replication of the forward with the spot.

When used to price a Cap, for example, the underlying forward rates of the Cap are thus assumed to be lognormal. Similarly, when used to price a Swaption (an option on a Swap), the underlying Swap rate is assumed to be lognormal.

The log-normality can be justified when pricing Cap/Floors and Swaptions independently (Jamshidan (1996), Miltersen, Sandmann, and Sondermann (1997)). Still, a simultaneous valuation of both a Cap and a Swaption with the Black formula is theoretically inconsistent. Both the forward rate of the Cap and the Swap rate cannot be lognormal simultaneously. However, the great popularity of this model for pricing both Caps and Swaptions indicates that any problems due to this inconsistency are negligible in an economic sense. Traders use to adjust this inconsistency by adjusting the volatility based on experience for the particular market in which they operate.

The same is true with bond prices and Swap rates; they cannot be lognormal at the same time. For instance, if the bond price is assumed to be lognormal, the continuously compounded Swap rate must be normally distributed.

19.1 Par and Forward Volatilities

Volatilities quoted on the market are par volatilities applied to some generic instruments. As an example, we see below the Cap volatilities based on 3month USD Libor.

Cap maturity	Volatility [%]
1 yr	10.37
2 yr	12.87
3 yr	14.12
4 yr	15.12
5 yr	15.25
7 yr	15.13
10 yr	14.88

The par or average volatility of 10.37 % would apply for all the three Caplets in a 1 year Cap (normally, there is no option on the first Libor fixing), the par volatility of 12.87 % would apply to all seven Caplets in a 2 year Cap, and so on.

This makes the quotation in terms of volatility very easy. However, the first three Caplets in the 2-year Cap must be identical to the Caplets in the 1-year Cap. Therefore, it would seem sensible that they should always be priced using the same volatilities.

Let us define "forward" volatility[32] as the volatility that would apply to a single Caplet. The forward volatility for the very first Caplet would be the volatility of a 3-month rate, which will be fixed in 3 months' time. The forward volatility for the second Caplet would be the volatility again of a 3-month rate, but this time fixed in 6 months' time, and so on.

In the Cap market, forward volatilities are derived from quoted par volatilities. Let T denote the maturity of the Tth generic Cap for which we have par volatilities. Define the price of a Cap of maturity T using par volatility V_T as:

$$C_T = \sum_{t \le T} c_t(V_T)$$

[32] Some does not use the confusing, expression ‘forward volatility’. Since volatility is not a traded asset, it can therefore be either present or future, but not forward. Similar, one might not use the term ‘forward-forward volatility’, which probably is used to mean, the future volatility of a forward quantity.

where $c_t(V_T)$ is the price of a single Caplet of maturity t. For arbitrage reasons, the same Cap using the forward volatility curve should have the same price:

$$C_T = \sum_{t \le T} c_t(V_T) = \sum_{t \le T} c_t(v_t)$$

where v, is the single period forward volatility. Hence, we can define a recursive relationship:

$$C_T = C_{T-1} = \sum_{T-1 \le t \le T} c_t(v_t)$$

A crude but common assumption is to set v, equal to a constant for T — $1 < t < T$. Then we can calculate sequentially the forward volatilities. Also, remember that par volatilities are most appropriate for ATM options.

To estimate the forward volatility curve we use the following process:

1. Guess a forward piece-wise constant volatility curve.
2. Price each of the Caps using this curve.
3. Adjust each segment of the volatility curve, starting at the short end in a bootstrapping fashion, so that the price of each Cap based on the forward volatility curve matches the original price.

We end up with a curve like Figure 19-1.

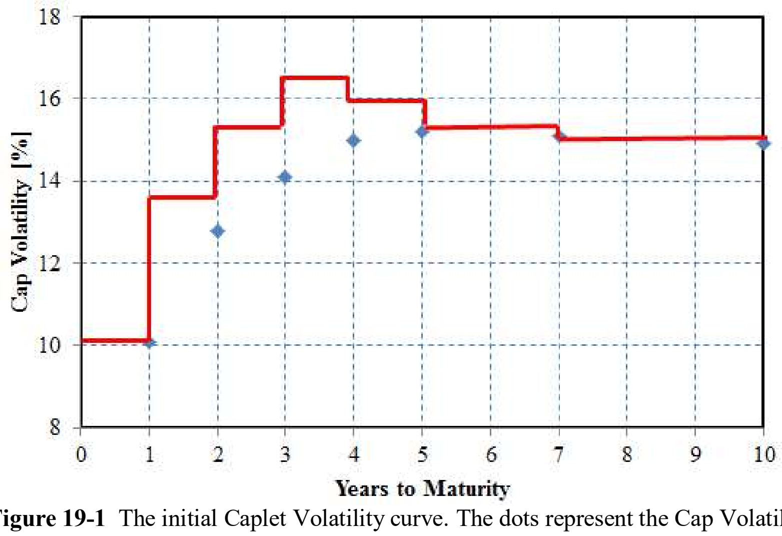

Figure 19-1 The initial Caplet Volatility curve. The dots represent the Cap Volatility

Whilst such a curve is arbitrage free, a smoother curve would be better. The approach may use an optimization technique using a smoothness criterion:

$\sum(\sigma_t - \sigma_{t-1})^2$ which has to be minimized whilst still retaining the arbitrage-freeness.

The result might looks like

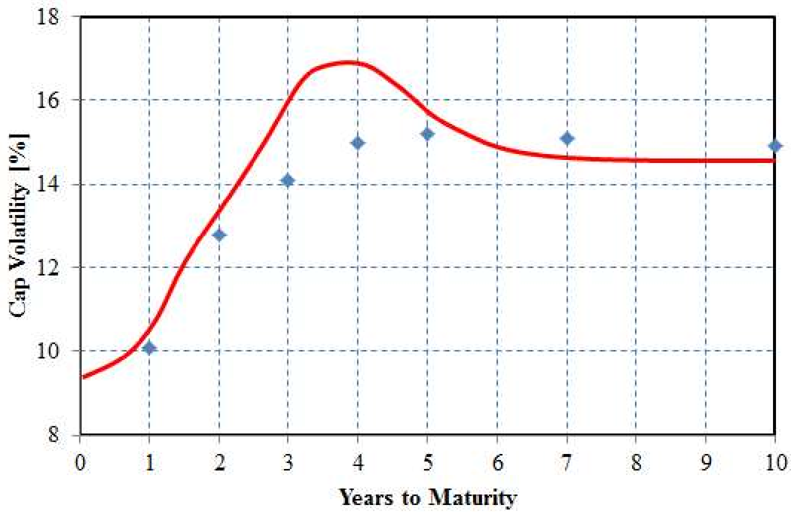

Figure 19-2 The optimized bootstrapped Caplet Volatility

Notice the very typical "humped" structure over the 2-5 year region; this is likely because of the traditional high demand by end-users for interest rate protection over those maturities.

Remark that the volatilities cannot be linear interpolated with respect to time. Instead, you need to interpolate the squared volatility multiplied with time. This gives us the following interpolation formula

$$\sigma^2(t)\cdot t = \sigma^2(T_1)\cdot T_1 + \frac{\sigma^2(T_2)\cdot T_2 - \sigma^2(T_1)\cdot T_1}{T_2 - T_1}(t - T_1)$$

giving

$$\sigma(t) = \sqrt{\sigma^2(T_1)\cdot\frac{T_1}{t} + \frac{\sigma^2(T_2)\cdot T_2 - \sigma^2(T_1)\cdot T_1}{T_2 - T_1}(1 - \frac{T_1}{t})}$$

These curves are often combined with statistical confidence bands. In practice it is found that volatilities do revert to a long-run level (as suggested by the ARCH model), which means that the confidence bands are wider at the short end than at the longer end. The bands are often called "volatility cones" due to their shape, and are used by traders to imply the likely movement of volatility through time.

We have just derived forward volatilities from a single ATM par volatility curve. It is however, common practice to use volatility surfaces, i.e. a matrix of strike vs. forward start date, when pricing and valuing Caps and Floors. This allows the smile effect to be incorporated. IR options on 3 month Libor are the most common, probably reflecting the fact that one can get exchange-traded options on 3-month deposit futures for hedging. Therefore, the most liquid volatility surface would also be on 3 month Libor, and volatility surfaces for other tenors represented by an offset surface from the 3-month one. A more complete approach therefore would be to model the entire two-dimensional surface. This surface is likely to contain gaps due to missing maturities and also missing volatilities for particular strikes.

19.2 Caps and Floors

As we have seen, an interest-rate Cap consists of a series of individual European call options, called Caplets. Each Caplet can be priced by using a modified version of the Black-76 formula. This is accomplished by using the implied forward rate, *F*, at each Caplet maturity as the underlying asset. The price of the Cap is the sum of the price of the Caplets that make up the Cap. Similarly, the value of a Floor is the sum of the sequence of individual put options, called Floorlets that make up the Floor.

As we know, the Black formula is

$$P_{call} = e^{-rT}(F \cdot N(d_1) - K \cdot N(d_2))$$
$$P_{put} = e^{-rT}(K \cdot N(-d_2) - F \cdot N(-d_1))$$

where F is the forward price and

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Consider the pricing model of a Caplet whose ceiling rate is L_c. The holder of the Cap receive at time t_i an amount equal to $\alpha_i \max\{L_{i-1}(T_{i-1}) - L_c, 0\}$ .The present value of this payment at L_{i-1} is

$$\frac{\alpha_i}{1+\alpha_i L_{i-1}(T_{i-1})}\max\{L_{i-1}(T_{i-1}) - L_c, 0\} = \max\left\{1 - \frac{1+\alpha_i L_c}{1+\alpha_i L_{i-1}(T_{i-1})}, 0\right\}$$

Remember the value of a pure discount bond;

$$p(T_{i-1}, T_i) = \frac{1}{1+\alpha_i L_{i-1}(T_{i-1})}$$

Therefore, the quantity

$$\frac{1+\alpha_i L_c}{1+\alpha_i L_{i-1}(T_{i-1})}$$

can be considered as the value at time T_{i-1} of a discount bond that pays $1+\alpha_i L_c$ at time T. Hence, the payoff in the max{.} above is the same as that from a put option with expiration date T_{i-1} on a bond with maturity time T_i. The par value of the bond is $1+\alpha_i L_c$ and the strike price of the put option is unity. Therefore, an interest rate Cap can be considered as a portfolio of European put options on discount bonds.

The time-t value of the Caplet can then be expressed as:

$$C_i(t,T_{i-1},T_i) = p(t,T_i) E^t_{Q_{T(i-1)}}\left[\alpha_i \max\left\{L_{i-1}(T_{i-1}) - L_c, 0\right\}\right]$$

Since $L_{i-1}(T_{i-1})$ is $\mathcal{F}_{T(i-1)}$-measurable, we may write

$$\begin{aligned} C_i(t,T_{i-1},T_i) &= p(t,T_{i-1}) E^t_{Q_{T(i-1)}}\left[p(T_{i-1},T_i)\alpha_i \max\left\{L_{i-1}(T_{i-1}) - L_c, 0\right\}\right] \\ &= p(t,T_{i-1}) E^t_{Q_{T(i-1)}}\left[\max\left\{1-\left(1+\alpha_i L_c\right) p(T_{i-1},T_i), 0\right\}\right] \end{aligned}$$

If we finally assume that the bond prices $p(t, T)$, under the risk neutral measure Q follow a general Gaussian process (see above), the time-t value of the Caplet is given by

$$C_i(t,T_{i-1},T_i) = p(t,T_{i-1})N(-d_2^{(i)}) - \left(1+\alpha_i L_c\right) p(t,T_{i-1})N(-d_1^{(i)}), \qquad t < T_{i-1}$$

where

$$d_2^{(i)} = \frac{\ln\left\{\dfrac{\left(1+\alpha_i L_c\right) p(t,T_i)}{p(t,T_{i-1})}\right\} - \dfrac{1}{2}\Sigma_i^2(t)(T_{i-1}-t)}{\Sigma_i(t)\sqrt{T_{i-1}-t}}$$

and

$$d_1 = d_2 + \Sigma_i(t)\sqrt{T_{i-1}-t}$$

and

$$\Sigma_i^2(t) = \frac{1}{T_{i-1}-t}\int_t^{T_{i-1}}\int_{T_{i-1}}^{T_i}\sum_{j=1}^{m}\sigma_F^2(s,u)\,du\,ds$$

The analytical expression above is complicated. This is because we have based the model on un-observable instantaneous forward rates. This makes the model difficult to implement and calibrate the volatility to market Cap data. This motivates to use a Market Model.

We therefore assume that the underlying forward LIBOR process is log-normal distributed with zero drift under some "market probability" Q_m. In its simples form we let the volatility denote the constant Black volatility of the forward LIBOR process

$$dL_{i-1}(t) = L_{i-1}(t)\sigma_{i-1}^{L} dW_t^m$$

where W_t^m is a Brownian process under Q_m. The Black formula for the time-t value of the Caplet that pays $\alpha_i \max\{L_{i-1}(T_{i-1}) - L_c, 0\}$ at time T_i is given by

$$\begin{aligned} C_i^{Black}(t, T_{i-1}, T_i) &= \alpha_i p(t, T_i) E_{Q_m}^t \left[\max\{L_{i-1}(T_{i-1}) - L_c, 0\} \right] \\ &= \alpha_i p(t, T_i) \left[L_{i-1}(t) N(d_1^{i-1}) - L_c N(d_2^{i-1}) \right] \end{aligned}$$

where

$$d_1^{i-1} = \frac{\ln\left\{\dfrac{L_{i-1}(t)}{L_c}\right\} - \dfrac{1}{2}\left(\sigma_{i-1}^{L}\right)^2 (T_{i-1} - t)}{\sigma_{i-1}^{L}\sqrt{T_{i-1} - t}}$$

and

$$d_2 = d_1 - \sigma_{i-1}^{L}\sqrt{T_{i-1} - t}$$

We can simplify this as

$$C(t) = \frac{N \cdot \tau}{1 + F \cdot \tau} e^{-r(T-t)} \left[F \cdot N(d_1) - K \cdot N(d_2) \right]$$

where τ is the tenor, N the face value and F the implied forward rate between time t and at the Caplets maturity, T. Similarly, for a Floorlet, we have

$$F(t) = \frac{N \cdot \tau}{1 + F \cdot \tau} e^{-r(T-t)} \left[K \cdot N(-d_2) - F \cdot N(-d_1) \right]$$

Example 19.1 We will illustrate the Cap value in a simple example. Suppose we have a Caplet, with six months to expiry on a 182-day forward rate and a face value of 100 million. The six-month forward rate is 8% (with act/360 as day-count), the strike is 8%, the risk-free interest rate 7%, and the volatility of the forward rate 28% per annum.

$F = 0.08$, $K = 0.08$, $T = 0.5$, $r = 0.07$, $\sigma = 0.28$.

$$d_1 = \frac{\ln(0.08/0.08) + (0.28^2/2)0.5}{0.28\sqrt{0.5}} = 0.0990, \quad d_2 = d_1 - 0.28\sqrt{0.5} = -0.0990$$

$$N(d_1) = 0.5394, \qquad N(d_2) = 0.4606$$

$$C(t) = \frac{10^9 \cdot \frac{182}{360}}{1 + 0.08 \cdot \frac{182}{360}} e^{-0.07 \cdot 0.5} \left[0.08 \cdot N(d_1) - 0.08 \cdot N(d_2)\right] = 295.995$$

19.3 Swaps and Swaptions

It is usual to distinguish between the two different types of Swaptions:

- Payer Swaptions. The right but not the obligation to pay fixed rate and receive floating rate in the underlying Swap.
- Receiver Swaptions. The right but not the obligation to receive fixed rate and pay floating rate in the underlying Swap.

Most Swaptions (about 90 %) is of European types and are normally priced by using the forward Swap rate as input in the Black-76 option-pricing model. The Black-76 value is multiplied by a factor adjusting for the tenor of the Swaption, as shown by Smith (1991). This is the practitioner's benchmark Swaption model. As illustrated by Jamshidan (1996), the model is arbitrage-free under the assumption of a lognormal Swap rate.

To derive a formula for a Swaption we will start by studying a forward starting Swap. That is a Swap that starts at a future time where we exchange floating against fixed cash flows. A $T_n \times (T_N - T_n)$ Swap means a Swap that starts at time T_n and have maturity at time T_N.

Denote the reset days for any Swap as: $T_0,\ T_1, T_N$ and define α_i as $T_i - T_{i-1}$. The holder of a forward starting $T_n \times (T_N - T_n)$ payer Swap with tenor T_N - T_n receives fixed payments at times $T_{n+1},\ T_{n+2}, ..., T_N$ and pay at the same times floating payments.

For each period $[T_i,\ T_{i+1}]$ the LIBOR rate $L_{i+1}(T_i)$ is set at time T_i and the floating leg $\alpha_{i+1} \cdot L_{i+1}(T_i)$ is received at T_{i+1}. For the same period the fixed leg $\alpha_{i+1} \cdot F$ is paid at T_{i+1} where F is the (fixed) Swap rate.

The arbitrage free value at $t < T_n$ of the floating payment made at T_i is given by $p(t,\ T_i)$ - $p(t,\ T_{i+1})$. The total value of the floating legs at time t for $t \leq T_n$ equals

$$\sum_{i=n+1}^{N} \alpha_i \cdot f(t,T_i)\cdot p(t,T_{i-1}) = \sum_{i=n}^{N-1} \alpha_{i+1} \cdot \frac{1}{\alpha_{i+1}} \frac{p(t,T_i) - p(t,T_{i+1})}{p(t,T_i)} \cdot p(t,T_i)$$
$$= \sum_{i=n}^{N-1} \left[p(t,T_i) - p(t,T_{i+1}) \right] = p(t,T_n) - p(t,T_N)$$
$$= p_n(t) - p_N(t)$$

where we have used that the forward rate is given by

$$p(0,t_i) = p(0,t_{i-1}) \cdot \frac{1}{1+\alpha_i f(t_{i-1},t_i)} \quad \Rightarrow \quad f(t_{i-1},t_i) = \frac{1}{\alpha_i} \frac{p(0,t_i) - p(0,t_{i-1})}{p(0,t_i)}$$

If we go back to the FRN, we remember that the value at the starting day is the same as the face value = 1. In a Swap, we do not have any final payment of the face value. This gives the Swap value at the starting day $t = 0$, as $1 - p(0, T)$. Between to resets we therefore must have the Swap value as: $p(t, t_0) - p(t, T)$ where t_0 is the time for the next reset day. This explains the formula above.

The total value at time t for the fixed side equals

$$\sum_{i=n}^{N-1} F \cdot p(t,T_{i+1})\alpha_{i+1} = F \sum_{i=n+1}^{N} \alpha_i p_i(t)$$

where F is called the Swap rate. This is a **par rate** since it makes the price of the Swap to be equal zero when entering the Swap contract. Therefore, the total value of the payer Swap is given by

$$PS_n^N(t,F) = p_n(t) - p_N(t) - F \sum_{i=n+1}^{N} \alpha_i p_i(t)$$

We therefore define the **forward Swap rate** (at par) $R_n^N(t)$ of the $T_n \times (T_N - T_n)$ Swap as the value of F for which the total value above is zero. I.e.,

$$R_n^N(t) = F = \frac{p_n(t) - p_N(t)}{\sum_{i=n+1}^{N} \alpha_i p_i(t)}$$

Therefore we also define for each pair n, k with $n < k$, the process

$$S_n^k(t) = \sum_{i=n+1}^{N} \alpha_i p_i(t)$$

as the **accrual factor** or **the value of a basis point** (also called the *level, DV01* Dollar Value change in a shift, *PV0l* Present Value change in a shift, *annuity* or *numerical duration* of the Swap).

We then express the Swap value as

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{S_n^N(t)}$$

In the market there are no quoted prices for different Swaps. Instead there are market quotes for the par Swap rates. We see that we can easily compute the arbitrage free price for a payer Swap with the strike rate K as

$$PS_n^N(t, R_n^N(t), K) = \left(R_n^N(t) - K\right) S_n^N(t)$$

A **payer Swaption** is then a contract given by;

$$P_n^N(t, R_n^N(T_n), K) = \max\left(R_n^N(T_n) - K, 0\right) S_n^N(T_n)$$

This contract gives the holder the right to enter a Swap contract at time T_n with Swaption strike (fixed rate) K.

Under the numeraire process S_n^N a payer Swaption is then a call option on R_n^N with strike price K. The value of this contract is given by the Black-76 formula:

$$P_n^N(t) = S_n^N(t)\left\{R_n^N(t)N(d_1) - KN(d_2)\right\}$$

where

$$d_1 = \frac{\ln\left\{\frac{R_n^N(t)}{K}\right\} + \frac{1}{2}\sigma_{n,N}^2\left(T_n - t\right)}{\sigma_{n,N}\sqrt{T_n - t}},$$

$$d_2 = d_1 - \sigma_{n,N}\sqrt{T_n - t}$$

The constant $\sigma_{n,N}$ is known as the **Black volatility**. Given a market price for a Swaption, the Black volatility implied by the Black formula is referred as the **implied Black Volatility**.

We can also write the Black formula as

$$P_n^N(t) = S_n^N(t)\left\{R_n^N(t)N(d_1) - K\cdot N(d_2)\right\} \equiv \sum_{i=n+1}^{N} \alpha_i p_i(t)\cdot\left\{R_n^N(t)\cdot N(d_1) - K\cdot N(d_2)\right\}$$
$$= \phi(t)\cdot\left\{F\cdot N(d_1) - K\cdot N(d_2)\right\}$$

or

$$P_n^N(t) = \frac{p_n(t) - p_N(t)}{R_n^N(t)} \left\{ R_n^N(t) \cdot N(d_1) - K \cdot N(d_2) \right\}$$

Here the function $\phi(t)$ is a discount function. If we denote the Forward Swap-rate between t_n and t_N as F, we have at t_n

$$\begin{aligned} p_n(t) - p_N(t) &\equiv p(t,t_n) - p(t,t_N) = p(t,t_n)\{1 - p(t_n,t_N)\} \\ &= p(t,t_n)\left\{1 - \frac{1}{(1+f(t_n,t_N))^{t_N - t_n}}\right\} \equiv p(t,t_n)\left\{1 - \frac{1}{(1+F)^{t_N - t_n}}\right\} \end{aligned}$$

If we now let $T = t_n$ be the maturity of the Swaption, F the Forward Swap-rate ($R_n^N(t)$ above) and introducing m reset days per year (the frequency) we finally have, where PS denote a payer Swaption, and RS is a receiver Swaption

$$PS = \frac{1 - \dfrac{1}{(1+F/m)^{\tau \cdot m}}}{F} e^{-rT} \left[F \cdot N(d_1) - K \cdot N(d_2) \right]$$

$$RS = \frac{1 - \dfrac{1}{(1+F/m)^{\tau \cdot m}}}{F} e^{-rT} \left[K \cdot N(-d_2) - F \cdot N(-d_1) \right]$$

$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2 \cdot T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where

τ = Tenor of Swap in years (time between Swaption maturity and Swap maturity).
F = Forward rate of the underlying Swap.
K = Strike rate of the Swaption.
r = Risk-free interest rate.
T = Time to Swaption expiration in years.
σ = Volatility of the forward-starting Swap rate.
m = Compounding's per year in Swap rate.

We also used continuous compounding, i.e., $p(t, T) = e^{-r(T-t)}$

Example 19.2 Consider a two-year payer Swaption on a four-year Swap with semi-annual compounding. The forward Swap rate of 7% starts two years from now and ends six years from now. The strike is 7.5%, the risk-free interest rate is 6%, and the volatility of the forward starting Swap rate is 20% per annum.

$$\tau = 4.0,\ m = 2,\ F = 0.07,\ K = 0.075,\ T = 2,\ r = 0.06,\ \sigma = 0.20.$$

$$d_1 = \frac{\ln(0.07/0.075) + (0.20^2/2)\cdot 2}{0.20\sqrt{2}} = -0.1025, \quad d_2 = d_1 - 0.20\sqrt{2} = -0.3853$$

$$N(d_1) = 0.4592, \qquad N(d_2) = 0.3500$$

$$c = e^{-0.06\cdot 2}\left[0.07\cdot N(d_1) - 0.075\cdot N(d_2)\right] = 0.5227\%$$

With a semi-annual forward Swap rate, the up-front value of the payer Swaption in per cent of the notional is

$$c\cdot\left[\frac{1-\frac{1}{(1+0.07/2)^{4\cdot 2}}}{0.07}\right] = 1.7964\%$$

19.3.1 The Greeks

If we return to the Black formulas we can first see that the Greeks are more complicated to calculate than the Greeks in the Black-Scholes formula. Delta for instance can be defined as the derivative of the *Premium* (Swaption value) with respect to the forward rate *F*. Other definitions of delta, is as the derivative with respect to the present value of the fixed leg of the Swap or as the derivative with respect to the annuity. We can also calculate a delta by shifting the yield-curve.

Let's try to explain the general difficulties when defining a delta; Options (including Swaptions) pricing is based on *models*. Those models have *parameters*. The market, on the other hand, takes this in the opposite direction. There are market prices and the model parameters are selected (calibrated) to achieve the market prices. In this context all the models provide the same (market) prices.

Regarding *delta,* the situation is different. The figures are not calibrated, they are the consequences of the price calibration and intrinsic model dynamic for the rates. You can therefore refer to two different types of deltas. The *theoretical* delta, which is a ratio. It is often referred to (in particular by Rebonato) as *in-the-model* delta (or hedging). The DV01 is obtained by shifting one rate (or the entire curve) by one basis point. This is (often) incompatible with the model used for the pricing. For that reason it is known as *out-of-the-model* delta (hedging).

In the formula

$$P_n^N(t) = \phi(t) \cdot \{F \cdot N(d_1) - K \cdot N(d_2)\}$$

we have seen that there is a "hidden" F. Some books and articles give delta as:

$$\Delta = \phi(t) \cdot N(d_1)$$

alternatively,

$$\Delta = N(d_1)$$

However, by making the approximation:

$$\frac{\Delta P_n^N(t)}{\Delta F} = \frac{P_n^N(t, F + dF) - P_n^N(t, F - dF)}{2 \cdot dF}$$

it is easy to prove that none of the deltas above gives the correct value. The calculation of delta is quite messy. Remember the calculation of delta for a stock option. We know that d_1 and d_2 also are functions of F. For the stock option d_1 and d_2 also includes the risk free rate.

To calculation of delta as the derivative with respect to F, we use:

$$PS = \frac{1 - \dfrac{1}{\left(1 + F/m\right)^{\tau \cdot m}}}{F} e^{-rT} \left[F \cdot N(d_1) - K \cdot N(d_2)\right]$$

and

$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2 \cdot T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

First, we have

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F} = \frac{1}{F \cdot \sigma \cdot \sqrt{T}}$$

then

$$\Delta = \frac{\partial PS}{\partial F} = \frac{\partial}{\partial F}\left\{\frac{1-\frac{1}{\left(1+F/m\right)^{\tau\cdot m}}}{F} e^{-rT}\left[F\cdot N(d_1) - K\cdot N(d_2)\right]\right\}$$

$$= e^{-rT}\frac{\partial}{\partial F}\left\{f(F)\cdot\left[N(d_1) - \frac{K}{F}\cdot N(d_2)\right]\right\} \quad \text{where} \quad \left\{f(F) = 1 - \frac{1}{\left(1+F/m\right)^{\tau\cdot m}}\right\}$$

$$= e^{-rT}\cdot\left\{\left[N(d_1) - \frac{K}{F}\cdot N(d_2)\right]\frac{\partial f(F)}{\partial F} + f(F)\cdot\frac{\partial}{\partial F}\left[N(d_1) - \frac{K}{F}\cdot N(d_2)\right]\right\}$$

We now have two derivatives, and they are calculated as

$$\frac{\partial}{\partial F}\left[N(d_1) - \frac{K}{F}\cdot N(d_2)\right] = \frac{\partial N(d_1)}{\partial F} + \frac{K}{F^2}N(d_2) - \frac{K}{F}\frac{F}{K}\frac{\partial N(d_1)}{\partial F} = \frac{K}{F^2}N(d_2)$$

and

$$\frac{\partial f(F)}{\partial F} = \frac{\partial}{\partial F}\left\{-\left(1+F/m\right)^{-\tau\cdot m}\right\} = \frac{\tau}{\left(1+F/m\right)^{\tau\cdot m}}$$

So

$$\Delta = e^{-rT}\left\{\left(1 - \frac{1}{\left(1+F/m\right)^{\tau\cdot m}}\right)\frac{K}{F^2}N(d_2) + \frac{\tau}{\left(1+F/m\right)^{\tau\cdot m+1}}\left[N(d_1) - \frac{K}{F}\cdot N(d_2)\right]\right\}$$

where we used

$$\frac{\partial N(d(F))}{\partial F} = \frac{\partial}{\partial F}\int_{-\infty}^{d(F)}\varphi(x)dx = \frac{\partial}{\partial F}\left[\Phi(x)\right]_{-\infty}^{d(F)} = \frac{\partial d(F)}{\partial F}\left[\varphi(d(F)) - \varphi(-\infty)\right]$$

$$= \frac{\partial d(F)}{\partial F}\varphi(d(F)) = \frac{\partial d(F)}{\partial F}N'(d(F)) = \frac{1}{F\sigma\sqrt{T}}N'(d(F))$$

and

$$N'(d_2) = \frac{\partial}{\partial F} N(d_1 - \sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}} \frac{\partial d_1}{\partial F} e^{-\left(d_1 - \sigma\sqrt{T}\right)^2/2}$$
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{F\sigma\sqrt{T}} e^{-(d_1)^2/2} e^{d_1\sigma\sqrt{T}} e^{-\sigma^2 T/2} = e^{d_1\sigma\sqrt{T}} e^{-\sigma^2 T/2} N'(d_1)$$
$$= e^{\ln(F/K) + \sigma^2 T/2} e^{-\sigma^2 T/2} N'(d_1) = \frac{F}{K} N'(d_1)$$

19.3.1.1 Greeks in the Black Model

If we ignore the annuity the Greeks in the Black model is given by:

$$\Delta_{call} = \frac{\partial C}{\partial F} = e^{-r(T-t)} N(d_1)$$
$$\Delta_{put} = \frac{\partial P}{\partial F} = e^{-r(T-t)} \left(N(d_1) - 1\right)$$

$$\Gamma = \frac{\partial^2 C}{\partial F^2} = \frac{\partial^2 P}{\partial F^2} = \frac{e^{-r(T-t)}}{F\sigma\sqrt{T-t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2}$$
$$\nu = \frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = F \cdot e^{-r(T-t)} \frac{\sqrt{T-t}}{\sqrt{2\pi}} \cdot e^{-d_1^2/2}$$

$$\Theta_{call} = \frac{\partial C}{\partial t} = e^{-r(T-t)} \left(r \cdot F \cdot N(d_1) - r \cdot K \cdot N(d_2) - \frac{F \cdot N'(d_1) \cdot \sigma}{2\sqrt{T-t}} \right)$$
$$\Theta_{put} = \frac{\partial P}{\partial t} = -e^{-r(T-t)} \left(r \cdot F \cdot N(-d_1) - r \cdot K \cdot N(-d_2) + \frac{F \cdot N'(d_1) \cdot \sigma}{2\sqrt{T-t}} \right)$$

$$\rho_{call} = \frac{\partial C}{\partial r} = t \cdot K \cdot e^{-r(T-t)} N(d_2)$$
$$\rho_{put} = \frac{\partial P}{\partial r} = -t \cdot K \cdot e^{-r(T-t)} N(-d_2)$$

19.4 Swaps in the multiple curve Framework

We saw above how to derive the Swap-rate for a forward starting Swap. If we generalize this for an ordinary Swap, under the multiple curve framework we generate the cash-flows with one curve (on tenor) and discount with another. Here we will study the difference when using one or two curves.

Denote the reset days for any Swap as: T_0, T_1, T_N and define α_i as the time interval $T_i - T_{i-1}$. The holder payer Swap with tenor $T_N - T_0$ receives fixed payments at times T_1, $T_2, \ldots, T_N$ and pay at the same times floating payments.

For each period $[T_i, T_{i+1}]$ the LIBOR rate $L_{i+1}(T_i)$ is set at time T_i and the floating leg $\alpha_{i+1} \cdot L_{i+1}(T_i)$ is received at T_{i+1}. For the same period the fixed leg $\alpha_{i+1} \cdot F$ is paid at T_{i+1} where F is the (fixed) Swap rate.

The arbitrage free value at $0 = t < T_n$ of the floating payment made at T_i is given by $p(T_i)$ - $p(T_{i+1})$. The total value of the floating legs at time t for $t \leq T_n$ equals

$$\begin{aligned}\sum_{i=1}^{N} \alpha_i \cdot f(T_{i-1}, T_i) \cdot p(T_i) &= \sum_{i=1}^{N} \alpha_i \cdot \frac{1}{\alpha_i} \frac{p(T_{i-1}) - p(T_i)}{p(T_i)} \cdot p(T_i) \\ &= \sum_{i=1}^{N} \left[p(T_{i-1}) - p(T_i) \right] = p(0) - p(T_N) \\ &= 1 - p(T)\end{aligned}$$

where we have used that the forward rate is given by

$$\begin{aligned} p(t_i) = p(t_{i-1}) \cdot p(t_{i-1}, t_i) \quad &\Rightarrow \quad p(t_{i-1}) \cdot \frac{1}{1 + \alpha_i f(t_{i-1}, t_i)} \\ \Rightarrow \quad f(t_{i-1}, t_i) = \frac{1}{\alpha_i} \frac{p(t_i) - p(t_{i-1})}{p(t_i)} & \end{aligned}$$

In the above analysis the forward rate $f(t, T_i)$ and the discount factor, $p(t, T_i)$ is given by the same curve/tenor. In a multi-curve framework we might generate the cash-flows with one curve (as below, a 3-month tenor curve) and discount with another (as below a 6-month tenor curve). Then, we have to modify the calculation as follows

$$\sum_{i=1}^{N} \alpha_i \cdot f_{3M}(T_{i-1}, T_i) \cdot p_{6M}(T_i) = \sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} \cdot p_{6M}(T_i)$$

We see that we cannot simplify this as we did when using the same tenors on both curves. The total value at time t for the fixed side, using a 6-month tenor for discounting equals

$$\sum_{i=1}^{N} F \cdot \alpha_i \cdot p_{6M}(T_i) = F \cdot \sum_{i=1}^{N} \alpha_i \cdot p_{6M}(T_i)$$

where F is the Swap rate. This is a **par rate** since it makes the price of the Swap to be equal zero when entering the Swap contract. So the total value of the payer Swap is given by

$$\begin{aligned} PS(F) &= \sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} \cdot p_{6M}(T_i) - F \cdot \sum_{i=1}^{N} \alpha_i \cdot p_{6M}(T_i) \\ &= \sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} - F \cdot \alpha_i \right) \cdot p_{6M}(T_i) \end{aligned}$$

With the old methodology, we should have the result

$$PS(F) = 1 - p(T) - F \sum_{i=1}^{N} \alpha_i \cdot p(T_i)$$

If we use the same tenors (a before the credit crises) for the cash-flow generation as for the discounting we derive the following Swap rate:

$$F = \frac{1 - p(T)}{\sum_{i=1}^{N} \alpha_i \cdot p(T_i)}$$

With different tenors we get

$$F = \frac{\sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} \cdot p_{6M}(T_i)}{\sum_{i=1}^{N} \alpha_i \cdot p_{6M}(T_i)}$$

So

$$\sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} - F \cdot \alpha_i \right) = 0$$

$$\Rightarrow$$

$$F \cdot T = \sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} = \sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1})}{p_{3M}(T_i)} - 1 \right)$$

and

$$F = \frac{1}{T} \sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1})}{p_{3M}(T_i)} - 1 \right)$$

19.5 Swaptions with forward premium

In the EURO, the Swaption market have changed to be traded with a forward premium in contrast to spot premium. In such way we can minimize the counterparty risk. Therefore we can also discount with the new risk-free interest rate, the EONIA Over Night Index-Swap Rate (OIS).

Say that we want to buy a payer Swaption at $t = 0$ with maturity at $t = T$. The Swap maturity here is denoted by $t = S$. The present value of the Swaption at $t = 0$ is in general given by

$$P(0,T,S) = S_T^S(0)\left\{F(0,T,S)\cdot N\left(d_1(0,T,S)\right) - K\cdot N\left(d_2(0,T,S)\right)\right\}$$

where $F(0, T, S)$ is the forward Swap rate between $t = T$ and $t = S$ contracted at $t = 0$ and

$$d_1(0,T,S) = \frac{\ln\left\{\frac{F(0,T,S)}{K}\right\} + \frac{1}{2}\sigma^2(0,T,S)\cdot T}{\sigma(0,T,S)\cdot\sqrt{T}},$$

$$d_2(0,T,S) = d_1 - \sigma(0,T,S)\cdot\sqrt{T}$$

and

$$S_T^S(0) = \sum_{i=T+\Delta t}^{S}\alpha_i p(t_i,S) = \frac{p(0,T)-p(0,S)}{F(0,T,S)}$$

Here $p(t, S)$ is the forward discount factor (zero coupon) between time t and S. This means that the premium at $t = 0$ is $P(0, T, S)$. If this is a forward premium, we shall at $t = T$ pay

$$premium = \frac{P(0,T,S)}{p(0,T)}$$

We therefore construct a portfolio consisting of the Swaption and the premium, so that the total value at $t = 0$ is zero. At any arbitrary time t, the value of our portfolio is

$$\begin{aligned} V(t,T,S) &= S_T^S(t)\left\{F(t,T,S)\cdot N\left(d_1(t,T,S)\right) - K\cdot N\left(d_2(t,T,S)\right)\right\} \\ &\quad - p(t,T)\cdot premium \\ &= \frac{p(t,T)-p(t,S)}{F(t,T,S)}\left\{F(t,T,S)\cdot N\left(d_1(t,T,S)\right) - K\cdot N\left(d_2(t,T,S)\right)\right\} \\ &\quad - p(t,T)\cdot premium \end{aligned}$$

where $F(t, T, S)$ is the forward Swap rate between $t = T$ and $t = S$ at time t and

$$d_1(t,T,S) = \frac{\ln\left\{\frac{F(t,T,S)}{K}\right\} + \frac{1}{2}\sigma^2(t,T,S)\cdot(T-t)}{\sigma(t,T,S)\cdot\sqrt{T-t}},$$

$$d_2(t,T,S) = d_1 - \sigma(t,T,S)\cdot\sqrt{T-t}$$

19.6 The Normal Black Model

Usually the underlying security is assumed to follow a lognormal process (or Geometric Brownian Motion). However, there are some traders who believe that the normal process describes the real market more closely than that of lognormal counterpart. This model is also known as the Bachelier's model.

Let us assume that the current future price, strike price, risk free interest rate, volatility, and time to maturity as denoted as f, K, r, σ, and $T - t$ respectively. Let us also assume that the current future price follows the following Normal process:

$$df = \mu dt + \sigma dW_t$$

where μ is a constant drift. For instruments like Swaptions, f represents the forward rate.

Let us start by studying the behaviour of the delta hedged portfolio which consists of long delta shares of future contract and short one derivative in question. Say, call it Π. Let us also denote the value of derivative by g. Then, the value of the delta-hedged portfolio is given by:

$$\Pi = g - \frac{\partial g}{\partial f} f$$

So applying Ito's lemma using the SDE given above into the changes of the portfolio value, one get

$$\begin{aligned} d\Pi &= dg - \frac{\partial g}{\partial f} df = \left(\frac{\partial g}{\partial t} + \mu\frac{\partial g}{\partial f} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial f^2}\right)dt + \sigma\frac{\partial g}{\partial f}dW - \frac{\partial g}{\partial f}\left(\mu dt + \sigma dW\right) \\ &= \left(\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial f^2}\right)dt \end{aligned}$$

We want the above quantity to be a Q-martingale under the discounted expectation with risk free rate. This is the same as stating that the above quantity equals the gain from the risk free interest rate for the portfolio value. So, we have:

$$d\Pi = r\Pi dt$$

Since it cost nothing to enter into a futures contract, one has: $\Pi = g$. Thus, we obtain the following PDE

$$\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial f^2} = rg$$

In a risk neutral world the process is giving as

$$df = \sigma dV_t$$

with the trivial solution, from integration over the interval $[t, T]$:

$$f(T) = f(t) + \sigma(V_T - V_t)$$

We see that f is a Gaussian process; $N[f_t, \sigma^2(T - t)]$, i.e., with mean $f(t)$ and variance $\sigma^2(T - t)$. By the application of Feynman-Kač, we obtain the following solution

$$\begin{aligned} g(t, f_T) &= e^{-r(T-t)} E^Q\left[\Phi(T)\right] = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \Phi(T) \cdot e^{-\frac{(f_T - f_t)^2}{2\sigma^2(T-t)}} df_T \\ &= \left\{ f_T = f_t + \sigma\sqrt{T-t} \cdot z \right\} \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \Phi(T) \cdot e^{-\frac{\left(\sigma\sqrt{T-t}\cdot z\right)^2}{2\sigma^2(T-t)}} \sigma\sqrt{T-t} dz \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(T) \cdot e^{-\frac{z^2}{2}} dz \end{aligned}$$

Here

$$\Phi(T) = \begin{cases} (f_T - K)^+ & \text{for a Call} \\ (K - f_T)^+ & \text{for a Put} \end{cases}$$

For the Call we have

$$\Pi_C(t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_T - K)^+ \cdot e^{-\frac{z^2}{2}} dz = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_t + \sigma\sqrt{T-t} \cdot z - K)^+ \cdot e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} (f_t + \sigma\sqrt{T-t} \cdot z - K) \cdot e^{-\frac{z^2}{2}} dz = A - B$$

Set $f_t = F$ and with $z_0 = \frac{(F-K)}{\sigma\sqrt{T-t}}$ we get

$$A = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} (F-K) \int_{z_0}^{\infty} e^{-\frac{z^2}{2}} dz = e^{-r(T-t)} (F-K) \cdot N[z_0]$$

$$B = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} \sigma\sqrt{T-t} \cdot z \cdot e^{-\frac{z^2}{2}} dz = e^{-r(T-t)} \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-\frac{z_0^2}{2}}$$

Then, the fair values of call C (payer Swaption, PS) and put P (receiver Swaption, RS) are expressed as

$$C = e^{-r(T-t)} \left[\left(F - K\right) \cdot N(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2} \right]$$

$$P = e^{-r(T-t)} \left[\left(K - F\right) \cdot N(-d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2} \right]$$

where

$$d = \frac{F-K}{\sigma\sqrt{T-t}}$$

and

- τ = Tenor of Swap in years (time between Swaption maturity and Swap maturity).
- F = Forward rate of the underlying Swap.
- K = Strike rate of the Swaption.
- r = Risk-free interest rate.
- T = Time to Swaption expiration in years.
- σ = Volatility of the forward-starting Swap rate.
- m = Compounding's per year in Swap rate.

To apply this on Swaptions we need, as above to multiply C and P with the annuity

$$\frac{1-\frac{1}{(1+F/m)^{\tau\cdot m}}}{F}$$

The Greeks can easily be calculated by simple differentiations

$$\Delta_C = \frac{\partial C}{\partial F} = e^{-r(T-t)} \cdot N(d)$$

$$\Delta_P = \frac{\partial P}{\partial F} = -e^{-r(T-t)} \cdot N(-d)$$

$$\Gamma = \frac{\partial^2 C}{\partial F^2} = \frac{\partial^2 P}{\partial F^2} = \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-d^2/2}$$

$$v = \frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = e^{-r(T-t)} \frac{\sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2}$$

$$\Theta_C = \frac{\partial C}{\partial t} = e^{-r(T-t)} \left(-r \cdot (F-K) \cdot N(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\cdot\pi}} e^{-d^2/2} - \frac{\sigma}{2\sqrt{2\cdot\pi\cdot t}} e^{-d^2/2} \right)$$

$$\Theta_P = \frac{\partial P}{\partial t} = e^{-r(T-t)} \left(r \cdot (F-K) \cdot N(-d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\cdot\pi}} e^{-d^2/2} - \frac{\sigma}{2\sqrt{2\cdot\pi\cdot t}} e^{-d^2/2} \right)$$

19.6.1 Convexity Adjustments

A standard bond or interest rate Swap has a convex price-yield relationship. To price options with the Black-76 model when the underlying asset is a derivative security with a payoff function linear in the bond or Swap yield, the yield should be adjusted for lack of convexity value.

Examples of derivatives where the payoff is a linear function of the bond or Swap yield are Constant Maturity Swaps (CMS) and Constant Maturity Treasury Swaps (CMT). The closed-form formula published by Brotherton-Ratcliffe and Iben (1993) assumes that the forward yield is lognormal distributed.

$$adj. = -\frac{1}{2}\frac{\frac{\partial^2 P}{\partial y_F^2}}{\frac{\partial P}{\partial y_F}} \cdot y_F^2 \left(e^{\sigma^2 T} - 1\right)$$

where

P = Bond or fixed side Swap value.
y_F = Forward yield.
T = Time to payment date in years.

σ = Volatility of the forward yield.

Example 19.3 Consider a derivative instrument with a single payment five years from now that is based on the notional principal times the yield of a standard four-year Swap with annual payments. The forward yield of the four-year Swap starting five years in the future and ending nine years in the future is 7%. The volatility of the forward Swap yield is 18%. Calculate the convexity adjustment of the Swap yield.

The value of the fixed side of the Swap with annual yield is equal to the value of a bond where the coupon is equal to the forward Swap rate/yield y_F.

$$P = \frac{c}{1+y_F} + \frac{c}{(1+y_F)^2} + \frac{c}{(1+y_F)^3} + \frac{1+c}{(1+y_F)^4}$$

The partial derivative of the Swap with respect to the yield is

$$\frac{\partial P}{\partial y_F} = -\frac{c}{(1+y_F)^2} - \frac{2c}{(1+y_F)^3} - \frac{3c}{(1+y_F)^4} - \frac{4(1+c)}{(1+y_F)^5}$$
$$= \{c = y_F = 0.07\} = -3.3872$$

and the second partial derivative with respect to the forward Swap rate is

$$\frac{\partial^2 P}{\partial y_F^2} = \frac{2c}{(1+y_F)^3} + \frac{6c}{(1+y_F)^4} + \frac{12c}{(1+y_F)^5} + \frac{20(1+c)}{(1+y_F)^6} = 15.2933$$

The convexity adjustment can now be found as:

$$adj. = -\frac{1}{2}\frac{15.2933}{-3.3872} \cdot 0.07^2 \left(e^{0.18^5} - 1\right) = 0.0019$$

The convexity-adjusted rate is then equal to 7.19% (0.07+0.0019).

19.6.2 Vega of the convexity adjustment

By taking the derivative of the convexity adjustment, we get the convexity adjustment's sensitivity to a small change in volatility

$$v = -\frac{\dfrac{\partial^2 P}{\partial y_F^2}}{\dfrac{\partial P}{\partial y_F}} \cdot y_F^2 \sigma T e^{\sigma^2 T}$$

19.6.3 Implied volatility from the convexity value in a bond

If the convexity adjustment is known, it is possible to calculate the implied volatility by simply rearranging the convexity adjustment formula

$$\sigma = \sqrt{\ln\left(\frac{adj.}{-\frac{1}{2}\left(\frac{\partial^2 P}{\partial y_F^2} \Big/ \frac{\partial P}{\partial y_F}\right) \cdot y_F^2} + 1\right)\frac{1}{T}}$$

19.7 European Short-Term Bond Options

European bond options can be priced in the Black-76 model by using the forward price of the bond at expiration as the underlying asset.

$$c = e^{-rT}(F \cdot N(d_1) - K \cdot N(d_2))$$
$$p = e^{-rT}(K \cdot N(-d_2) - F \cdot N(-d_1))$$

where F is the forward price of the bond at the expiration of the option, and

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

This model does not take into consideration the pull to par effect of the bond. At maturity, the bond price must be equal to principal plus the coupon. For this reason, the uncertainty of a bond will first increase and then decrease.

The Black-76 model assumes that the uncertainty (variance) of the underlying asset increases linearly with time to maturity. Pricing of European bond options using this approach should thus be limited to options with short time to maturity relative to the time to maturity of the bond. A rule of thumb used by some traders is that the time to maturity of the option should be no longer than one-fifth of the time to maturity on the underlying bond

Example 19.4 Consider a European put option with six months to expiry and strike price 122 on a bond with forward price at option expiration equal to 122.5. The volatility of the forward price is 4%, and the risk-free discount rate is 5%. Calculate the option's value.

$$F = 122.5,\ K = 122,\ T = 0.5,\ r = 0.05,\ \sigma = 0.04.$$

$$d_1 = \frac{\ln(122.5/122) - (0.04^2/2)\cdot 0.5}{0.04\sqrt{0.5}} = 0.1587, \quad d_2 = d_1 - 0.04\sqrt{0.5} = 0.1305$$

$$N(-d_1) = 0.4369, \qquad N(-d_2) = 0.4481$$

$$p = e^{-0.05\cdot 0.5}\left[122N(-d_2) - 122.5N(-d_1)\right] = 1.1155$$

19.8 The Schaefer and Schwartz model

Schaefer and Schwartz (1987) modified the Black-Scholes model for pricing bond options to takes into consideration that the price volatility of a bond increases with duration

$$c = S\cdot N(d_1) - Ke^{-rT}N(d_2)$$
$$p = Ke^{-rT}N(-d_2) - S\cdot N(-d_1)$$

where $\sigma = \left(\lambda S^{\alpha-1}\right)D$, and

$$d_2 = \frac{\ln(S/K) - (r+\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Here D is the duration of the bond after the option expires. λ is estimated from the observed price volatility σ_0 of the bond. α is a constant that Schaefer and Schwartz suggest should be set to 0.5.

$$\lambda = \frac{\sigma_0}{S^{\alpha-1}\cdot D^*}$$

where D^* is the duration of the bond today.

Example 19.5 Assume that the duration of the bond is eight years and that the observed price volatility of the bond is 12%. This gives: $\lambda = 0.15$.

In **Table 19-1** we uses this value and compares the option prices from the Schaefer and Schwartz formula with option prices from the Black-76 formula.

Table 19-1 Option prices from Schaefer and Schwartz and Black-76

Bond Duration	Base Volatility (%)	Adjusted Volatility (%)	Black-76 Value	Modified Black-76 Value
1	12.0	1.5	5.5364	0.6929
2	12.0	3.0	5.5364	1.3857
3	12.0	4.5	5.5364	2.0783
4	12.0	6.0	5.5364	2.7707
5	12.0	7.5	5.5364	3.4628
6	12.0	9.0	5.5364	4.1545
7	12.0	10.5	5.5364	4.8457
8	12.0	12.0	5.5364	5.5364