Chapter 18

Exotic Instruments

18 Some Exotic instruments

For some exotic instruments, we can use the Forward-Measure-Pricing described in the previous section. We will now describe methods of how we can calculate prices for such kind of derivatives.

18.1 Constant Maturity Contracts

Constant maturity contracts are instruments using a floating rate, based on a Swap index, i.e. the par rate of a generic Swap. They can be valued using the forward measure technology based on term structure models. This requires the mapping/calibration of a volatility structure with a term structure model.

Let the value of such a CMS contract be $g(R_f(T_1,T_2))$ at the payday T_p, where

$$R_f\left(T_1,T_2\right)=\frac{1}{\tau}\left[\frac{1-p\left(T_1,T_n\right)}{\sum_{i=2}^{n}p\left(T_1,T_i\right)}\right]$$

is the Swap rate, having $p(T_1,T_2)$ equal to a zero coupon bond price at T_1 of bond maturing at T_2. Here t is the reset period with T_1 as reset day and T_2 as payday. We will primarily study the calculations using the Hull & White model.

If the dynamics of the instantaneous rate under the measure Q is

$$dr = (\theta(t) - \kappa r)\, dt + \sigma\, dz,$$

and the forward measure Q^T is defined by

$$\frac{dQ^T}{dQ} = \frac{\exp\left\{-\int_0^T r(s)\,ds\right\}}{E^Q\left[\exp\left\{-\int_0^T r(s)\,ds\right\}\right]}$$

Then the present value of the contract $PV(t)$ can be expressed as

$$PV(t) = p(t,T_p)\,E^{Q_{T_p}}\left[g\left(R_f\left(T_p,T_s,T_e\right)\right)\middle| r(t) = r\right]$$

where the expectation value ends up in an integral in $r(T_p)$

$$PV(t) = p(t,T_p)\frac{1}{\sqrt{2\pi v}}\int_{-\infty}^{\infty} g(r)\exp\left(\frac{-(r-m)^2}{2v}\right)dr$$

with

$$m = E^{Q_{T_p}}\left[r(T_r)\right] = -\frac{\partial}{\partial s}\ln p(0,s)\Big|_{s=T_r} + \frac{\sigma^2}{2\kappa^2}e^{-\kappa T_r}\left[e^{-\kappa T_r} - e^{\kappa T_r} + e^{-\kappa T_p}e^{2\kappa T_r} - e^{-\kappa T_p}\right]$$

and

$$v = Var^{Q^{T_p}}\left[r(T_r)\right] = \frac{\sigma^2}{2\kappa}e^{-\kappa T_r}\left(e^{\kappa T_r} - e^{-\kappa T_r}\right)$$

If instead using the Ho & Lee model, we have

$$m = -\frac{\partial}{\partial s}\ln p(0,s)\Big|_{s=T_r} + \sigma^2\left(T_r^2 - T_p T_r\right)$$

and

$$v = \sigma^2 T_r$$

We can here use Romberg method to calculate the integral.

18.2 Compound Options

By compound options, we mean options on option-style instruments. These include:

- options on options
- options on Caps/Floors
- options on free defined cash flows where there is at least one optional cash flow.

To make an accurate valuation possible, these contracts must also be mapped/calibrated to a volatility structure with a term structure model. The underlying instrument should here use the same volatility structure as the compound option, regardless of how the underlying.

Compound options can be valued using the forward measure technique described under Constant Maturity Contracts. With this technique, the valuation is carried out by integrating the product of the payout function at expiry and a density function and then discounting the result.

For compound options, no Swap rate is involved and there is no need to calculate the expectation value of $r(T_r)$ under the forward measure Q^{Tp} (that is, T_r here equals T_p, which simplifies the calculations. The present value $PV(t)$ of a compound option is

$$PV(t) = p\left(t, T_p\right) \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} g(r) \exp\left(\frac{-(r-m)^2}{2v}\right) dr$$

with T_p = option expiration day,

$$g(r) = g\left(r\left(T_p\right)\right)$$

is the boundary condition, including the value at $r = r(T_p)$ of the underlying option/Cap/ Floor defined cash flow

$$m = E^{Q^{Tp}}\left[r\left(T_p\right)\right] = -\frac{\partial}{\partial s} \ln p(0, s)\bigg|_{s=T_p}$$

Using Hull & White

$$v = Var^{Q^{Tp}}\left[r\left(T_p\right)\right] = \frac{\sigma^2}{2\kappa}\left(1 - \exp\left(-2\kappa T_p\right)\right)$$

or Ho & Lee

$$v = \sigma^2 T_p$$

Romberg's method is used when calculating the integral.

18.3 Quanto Contracts

Quanto contracts have floating cash flows where the reference rate is a rate index in a currency other than the payout currency. Such quanto products are:

- Differential Swaps
- Quanto Caps/Floors
- Quanto bond options
- Swaptions

These products can be valued according to term structure models Ho & Lee or Hull & White.

The model is a multi-factor model in the sense that the domestic rate, $r(t)$, the foreign rate, $y(t)$, and the exchange rate are modelled as stochastic processes:

$$dr(t) = \alpha_r dt + \sigma_r dZ$$

and

$$dy(t) = \alpha_y dt + \sigma_y dW^F$$

For the exchange rate $S(t)$, we have the following differential equation in the domestic world

$$\frac{dS(t)}{S(t)} = (r - y)\,dt + \sigma_s dX$$

The T-forward exchange rate is

$$f_S(t) = \frac{S(t)q^T(t)}{p^T(t)}$$

For valuation, the correlation between the domestic interest rate and the foreign interest rate, ρ as well as the correlation between the foreign interest rate and the exchange rate, δ is needed.

In addition, a quanto option needs three volatility structures for the valuation. The volatility of the domestic rate of the option, the volatility of the foreign rate of the underlying asset and the volatility of the exchange rate of the underlying currency.

A quanto Swaption is defined by selecting a different instrument currency for the option than the underlying Swap currency. The type of quanto bond option that is valued as the difference between the price of the foreign bond, $q(T)$, and a fixed amount in the domestic currency (strike K):

$$payout = \max(q(T) - K, 0),$$

with the payout in the domestic currency.

The explicit formulas for each contract type are given below.

18.3.1 Differential Swaps

Differential Swaps are valued using the following formula:

$$PV(t) = p^{t1}(t)\left[\frac{1}{q^{t1}(t_0)} - \frac{1}{p^{t1}(t_0)}\right] + \sum_{i=2}^{n}\left(D_{t_{i-1},t_i}(t) - p^{t_{i-1}}(t)\right)$$

where

$p^T(t)$ is the value of a zero coupon bond paying out 1 unit of the domestic currency at time T,

$q^T(t)$ is the value of a zero coupon bond paying out 1 unit of the foreign currency at time T

$$D_{T,\tau}(t) = \frac{p^{\tau}(t)q^{T}(t)}{q^{\tau}(t)}a(t)$$

$a(t) = e^{\mathrm{cov}(t,T,\tau)}$ is the "correction factor" that takes the quanto effect into account and where cov(...) is model dependent.

In Ho & Lee we have

$$\mathrm{cov}(t,T,\tau) = \sigma_y(\tau - T)(T - t)\left[\left[\delta\sigma_s - \sigma_y\tau + \rho\sigma_r T\right] + \frac{\sigma_y - \rho\sigma_r}{2}(T + t)\right]$$

and in Hull & White we have

$$\mathrm{cov}(t,T,\tau) = C(T,\tau)\cdot\left[I_1 + I_2 + I_3\right]$$

where

$$C(T,\tau) = \frac{\sigma_y}{\kappa_y}\left[e^{-\kappa_y(\tau - T)} - 1\right]$$

$$I_1 = \frac{\delta\sigma_s}{\kappa_y}\left(1 - e^{-\kappa_y(T-t)}\right)$$

$$I_2 = \frac{\sigma_y}{\kappa_y}\left[\frac{1 - e^{-\kappa_y(T-t)}}{\kappa_y} - \frac{1 - e^{-2\kappa_y(T-t)}}{2\kappa_y}\right]$$

and

$$I_3 = \frac{\rho\sigma_r}{\kappa_r}\left[\frac{1 - e^{-\kappa_y(T-t)}}{\kappa_y} - \frac{1 - e^{-(\kappa_y + \kappa_r)(T-t)}}{\kappa_y + \kappa_r}\right]$$

δ is the correlation between the foreign interest rate and the exchanged rate and ρ is the correlation between the domestic and foreign short interest rates.

The time t_0 does not have to be the starting time of the contract, it can be any reset date.

18.3.1.1 Quanto Caps/Floors

Quanto Caps/Floors have a present value that equals the sum of the present value of each Caplet. The Caplet value is

$$Caplet(t) = p^T(t)\left[\frac{a(t)\cdot q^T(t)}{q^\tau(t)}\Phi(d_+) - \left(1 + (\tau - T)\cdot R_{CAP}\right)\Phi(d_-)\right]$$

where

$$\Phi(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}dx,$$

$$d_\pm = \frac{\ln\left(\dfrac{a(t)\cdot q^T(t)}{q^\tau(t)\cdot\left(1 + (\tau - T)\cdot R_{CAP}\right)}\right)}{v(t)} \pm \frac{v(t)}{2},$$

$$v^2(t) = \int_t^T \mathrm{var}_s\left[\frac{df(s)}{f(s)}\right]$$

and

$$f = \frac{a \cdot q^T}{q^t}$$

The rest of the variables used are the same as for differential Swaps.

18.3.1.2 Quanto Bond Options

According to a result obtained by Jamshidian, an option on a portfolio of zero-coupon bonds can be valued as a portfolio of options on zero-coupon bonds

$$C(t) = \max\left(0, q_{coupon}(T) - K\right) = \sum_{i=1}^{n} c_i \max\left(0, q^{T_i}(T) - D_i\right)$$

If all bond prices are continuously decreasing functions of the short interest rate y, there is a value of y where the value of the coupon bond equals the strike price K. Let us call this value $y*$.

The option will only be exercised if the value of the short rate is below $y*$. With $y*$ and the formula for bonds as a function of the interest rate, discount factors D_i can be calculated from the exercise date T to the different coupon dates T_i.

The value of the *ith* option on a zero coupon bond is then:

$$c_i \max\left(0, q^{T_i}(T) - D_i\right) = p^T(t)\left[\frac{q^{T_i}(t)a(t)}{q^T(T)}\Phi(d_+) - D_i\Phi(d_-)\right]$$

where the parameters are the same as above and

$$\Phi(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}dx$$

$$d_\pm(t) \equiv \frac{\ln\left(f(t)/D_i\right)}{v(t)} \pm \frac{v(t)}{2}$$

$$v^2(t) = \int_t^T \mathrm{var}_s\left[\frac{df(s)}{f(s)}\right]$$

and

$$f = \frac{q_i^T \cdot a}{q^T}$$