Chapter 17

A new Measure – The Forward Measure

17 Forward Measures

In previous sections, we have used two probability measures: **the objective (real) probability measure *P***, and the **"risk neutral" martingale measure *Q***. In this section we will introduce a whole new class of probability measures, so called forward measures, including Q as a member of that class. These probability measures are connected to a technique called **change of numeraire**. They are of great importance both in the understanding and for practical calculations since the amount of computational work needed in order to obtain a pricing formula can be drastically reduced by a suitable choice of numeraire. Especial will the forward measures simplify the calculations of prices on bond options.

To get some feeling for where we are heading, let us consider a pricing problem. But first we remember that a martingale is a zero-drift stochastic process. We will also in general think of measures as units in which we value other securities. If we use the price of a traded security as such a unit measure, then there is some market price of risk for which all other security prices are martingales.

Suppose that $p_1(t, T)$ and $p_2(t, T)$ are prices of two traded securities that depend on a single source of uncertainty. Define the relative price of $p_1(t, T)$ with respect to $p_2(t, T)$ as

$$\gamma = \frac{p_1(t,T)}{p_2(t,T)}$$

We refer $p_2(t, T)$ as the **numeraire**. The equivalent martingale measure states that, if there are no arbitrage opportunities, γ is martingale for some market price of risk. What is more, for a given numeraire security $p_2(t, T)$, the same choice of the market price of risk makes γ martingale for all securities $p_1(t, T)$. This choice of market price of risk is the volatility of $p_2(t, T)$. We can state this as a **theorem** and we now give a proof.

Proof: Suppose the volatilities of $p_1(t, T)$ and $p_2(t, T)$ are σ_1 and σ_2. In general, when we introduce the market price of risk, λ we have:

$$dp = (r + \lambda\sigma)pdt + \sigma pdW$$

Therefore

$$\begin{cases} dp_1 = (r + \sigma_1\sigma_2)p_1 dt + \sigma_1 p_1 dW \\ dp_2 = (r + \sigma_2^2)p_2 dt + \sigma_2 p_2 dW \end{cases}$$

Using Itô's lemma we get

$$\begin{aligned}
d\left(\frac{p_1}{p_2}\right) &= \frac{\partial}{\partial p_1}\left(\frac{p_1}{p_2}\right)dp_1 + \frac{\partial}{\partial p_2}\left(\frac{p_1}{p_2}\right)dp_2 + \frac{1}{2}\frac{\partial^2}{\partial p_2^2}\left(\frac{p_1}{p_2}\right)(dp_2)^2 \\
&+ \frac{\partial^2}{\partial p_1 \partial p_2}\left(\frac{p_1}{p_2}\right)dp_1 dp_2 \\
&= \frac{1}{p_2}\left\{(r + \sigma_1\sigma_2)p_1 dt + \sigma_1 p_1 dW\right\} - \frac{p_1}{p_2^2}\left\{(r + \sigma_2^2)p_2 dt + \sigma_2 p_2 dW\right\} \\
&+ \frac{1}{2}2\frac{p_1}{p_2^3}\sigma_2^2 p_2^2 dt - \frac{1}{p_2^2}\sigma_1\sigma_2 p_1 p_2 dt \\
&= \frac{p_1}{p_2}\left\{(r + \sigma_1\sigma_2)dt + \sigma_1 dW\right\} - \frac{p_1}{p_2}\left\{(r + \sigma_2^2)dt + \sigma_2 dW\right\} \\
&+ \frac{p_1}{p_2}\sigma_2^2 dt - \frac{p_1}{p_2}\sigma_1\sigma_2 dt \\
&= \frac{p_1}{p_2}\sigma_1 dW - \frac{p_1}{p_2}\sigma_2 dW = \frac{p_1}{p_2}(\sigma_1 - \sigma_2)dW
\end{aligned}$$

We then see that p_1/p_2 is martingale. We say that in a world where the market price of risk is σ_2 the world is **forward risk neutral** with respect to p_2. Therefore we call this measure (in terms of p_2) as a **forward measure**.

This very simple analysis shows that we can change to any numeraire security, use that as a forward measure where the market price of risk is the volatility of the numeraire security. Then in terms of this security all processes becomes martingales.

We have used exactly this in option pricing on equities, where we used $B(t)$ as the numeraire security. Then we found that the discounted stock price was martingale. If we remember the "roll-over strategy" in the bond pricing section, we called this the **money market account**.

Because f/g, where f and g are any securities, is martingale in a world that is forward risk neutral with respect to g, it follows that

$$\frac{f(t)}{g(t)} = E^g\left[\frac{f(s)}{g(s)} \mid \mathcal{F}_t\right]$$

or

$$f(t) = g(t) \cdot E^g\left[\frac{f(s)}{g(s)} \mid \mathcal{F}_t\right]$$

Let us now consider the pricing problem for a contingent claim X, in a model with a stochastic short rate of interest $r(t)$. From the general theory we know that the price at $t = 0$ of X is given by the formula

$$\Pi[0, X] = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\} \cdot X\right]$$

The problem with this formula from a computational point of view, is that, in order to compute the expected value we have to get hold of the joint distribution (under Q) of the two stochastic variables (the integral of $r(s)$ and X) and finally integrate with respect to that distribution. Thus we have to compute a double integral, and in most cases this turns out to be rather hard work.

Let us now make the (extremely unrealistic) assumption that r and X is independent under Q. Then the expectation above splits, and we have the formula

$$\Pi[0, X] = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\right] E^Q[X]$$

which we may write as

$$\Pi[0, X] = p(0, T) \cdot E^Q[X]$$

We now note that this is a much nicer formula since:

- We only have to compute the single integral $E^Q[X]$ instead of the double integral.
- The bond price $p(0, T)$ does not have to be computed theoretically at all. We can **observe** it (at $t = 0$) directly on the bond market.

The drawback with the argument above is that, in most concrete cases, r and X are not independent under Q, and if X is a contingent claim on an underlying bond, this is of course obvious. What may be less obvious is that even if X is a claim on an underlying stock that is P-independent of r, it will still be the case that X and r will be dependent (generically) under Q. The reason is that under Q the stock will have r as its local rate of return, thus introducing a Q-dependence.

This is the bad news. The good news is that there exists a **general** pricing formula, a special case of which reads as

$$\Pi[0,X] = p(0,T)\cdot E^T[X]$$

Here E^T denotes expectation with respect to the so-called **forward neutral measure** Q^T, which we will discuss below. We see from this formula that we do indeed have the multiplicative structure, but the price we have to pay for generality is that the measure Q^T depends upon the choice of maturity date T. We define, on the bond market, the forward measure Q^T on $\mathcal{F}^T$ as:

Definition 17.1 Let T be a fixed time. Then *the forward measure Q^T on $\mathcal{F}^T$* is defined by

$$\frac{dQ^T}{dQ} = \frac{\exp\left\{-\int_0^T r(s)ds\right\}}{E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\right]}$$

I.e. the Radon-Nikodym derivative R^T is given by

$$R^T = \frac{1}{p(0,T)}\exp\left\{-\int_0^T r(s)ds\right\}$$

It is very important to notice that we get different measures Q^T for different choices of T.

We will now prove a stronger pricing formula:

Theorem 17.2. *Let X be a given T-claim. Then the arbitrage free price of X is given by.*

$$\Pi[t,X] = p(t,T)\cdot E^T[X \mid \mathcal{F}_t]$$

where E^T quote integrations with respect to Q^T.

Proof: If we use

$$E^T\left[X \mid \mathcal{F}_t\right] = \frac{E^Q\left[R^T X \mid \mathcal{F}_t\right]}{E^Q\left[R^T \mid \mathcal{F}_t\right]}$$

We then have

$$\begin{aligned}
E^Q\left[R^T X \mid \mathcal{F}_t\right] &= \frac{1}{p(0,T)} E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\} X \mid \mathcal{F}_t\right] \\
&= \frac{1}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\} E^Q\left[\exp\left\{-\int_t^T r(s)ds\right\} X \mid \mathcal{F}_t\right] \\
&= \frac{\Pi\left[t, X\right]}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\}
\end{aligned}$$

and

$$E^Q\left[R^T \mid \mathcal{F}_t\right] = \frac{1}{p(0,T)} E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\} \mid \mathcal{F}_t\right] = \frac{p(t,T)}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\}$$

These two gives the result.

Theorem 17.3. *The likelihood process L^T is given by:*

$$L^T = \frac{p(t,T)}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\} = \frac{p(t,T)}{p(0,T)B(t)}$$

Proof: Since L^T is Q^T martingale

$$L^T(t) = E^Q\left[L^T(t) \mid \mathcal{F}_t\right] = E^Q\left[R^T \mid \mathcal{F}_t\right] = \frac{p(t,T)}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\} = \frac{p(t,T)}{p(0,T)B(t)}$$

If we are using one-factor models where all uncertainty is generated by the Q-Wiener process V, we know that all absolute continuous transformations of measure are given by a Girsanov kernel. Especially, there must exist a Girsanov transformation between Q and Q^T. We are curious of how this Girsanov transformation looks like.

We know that the dynamic of a T-bond under Q is given by

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dV(t)$$

With Itô we get

$$dL^T(t) = \frac{\partial L^T}{\partial p}dp + \frac{\partial L^T}{\partial B}dB = \frac{1}{p(0,T)B(t)}dp - \frac{p(t,T)}{p(0,T)B^2(t)}dB$$
$$= \frac{1}{p(0,T)B(t)}\{r(t)p(t,T)dt + \nu(t,T)p(t,T)dV(t)\} - \frac{p(t,T)}{p(0,T)B^2(t)}r(t)B(t)dt$$
$$= \frac{p(t,T)}{p(0,T)B(t)}\nu(t,T)dV(t) = \nu(t,T)L^T(t)dV(t)$$

Therefore we have proved the following result:

Theorem 17.4. *For a given T, the Girsanov transformation from Q to Q^T is given by a likelihood process given by:*

$$dL^T(t) = \nu(t,T)L^T(t)dV(t)$$

The Girsanov kernel is given by the process $\nu(t, T)$ *and the likelihood process* L^T *have the representation given by*

$$L^T(t) = \exp\left\{\int_0^t \nu(s,T)dV(s) - \frac{1}{2}\int_0^t \nu^2(s,T)ds\right\}$$

Furthermore

$$dV(t) = \nu(t,T)dt + dW^T(t)$$

where $dW^T(t)$ is a Q^T-Wiener process on the interval [0, T].

The forward measures have an important economical interpretation as well. In the above discussion we have used a *T*-bond as numeraire. But generally we can use any security as the numeraire process.

In order to understand why formulas of this type have anything to do with the choice of numeraire, let us give a very brief and informal argument.

We start by recalling that the risk neutral martingale measure *Q* has the property that for every choice of a price process $\Pi(t)$ for a traded asset, the quote:

$$\frac{\Pi(t)}{B(t)}$$

is a *Q*-martingale. The point here is that we have divided the asset price $\Pi(t)$ by the **numeraire asset price** $B(t)$. It is now natural to investigate whether this martingale property can be generalized to other choices of numeraire, and we are led to the following conjecture:

Consider a fixed financial market, and a fixed "numeraire" asset price process $S_0(t)$ on the market. Then there exists a probability measure, denoted Q^0, such that

$$\frac{\Pi(t)}{S_0(t)}$$

is a Q^0-martingale for every asset price process $\Pi(t)$.

Let us for the moment assume that the conjecture is true. We then fix a certain date of maturity T, and we choose the bond price process $p(t, T)$ (for this fixed T) as numeraire. According to the conjecture there should then exist a probability measure, which we denote by Q^T, such that the quotient

$$\frac{\Pi(t)}{p(t,T)}$$

is a martingale under the measure Q^T, for every $\Pi(t)$ which is the price process of a traded asset. In particular we have (using the relation $p(T, T) = 1$)

$$\frac{\Pi(0)}{p(0,T)} = E^T\left[\frac{\Pi(T)}{p(T,T)}\right] = E^T\left[\Pi(T)\right]$$

where E^T denotes expectation under Q^T. Let us now choose the derivative price process $\Pi(t, X)$ as $\Pi(t)$ above. Then we have $\Pi(T) = \Pi(t, X) = X$:

$$\frac{\Pi(0,X)}{p(0,T)} = E^T\left[X\right]$$

$\Rightarrow$

$$\Pi\left(0,X\right) = p(0,T)\cdot E^T\left[X\right]$$

An alternative is a try to construct a measure Q^T with $p(t, T)$ as numeraire that makes all prices to martingales. Especially this measure should make

$$Z^T(t) = \frac{B(t)}{p(t,T)}$$

to a martingale. If we use the dynamics of $p(t, T)$ under Q:

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dV(t)$$

and use Itô's formula on Z^T

$$
\begin{aligned}
dZ^T(t) &= \frac{\partial Z^T}{\partial B} dB + \frac{\partial Z^T}{\partial p} dp + \frac{1}{2}\frac{\partial^2 Z^T}{\partial p^2}\left(dp\right)^2 \\
&= \frac{1}{p(t,T)} dB - \frac{B(t)}{p^2(t,T)} dp + \frac{B(t)}{p^3(t,T)}\left(dp\right)^2 \\
&= \frac{B(t)}{p(t,T)} rdt - \frac{B(t)}{p^2(t,T)}\left(rp(t,T)dt + v(t,T)p(t,T)dV(t)\right) \\
&+ \frac{B(t)}{p^3(t,T)} v^2(t,T)p^2(t,T)dt \\
&= Z^T(t)rdt - Z^T(t)\left(rdt + v(t,T)dV(t)\right) + Z^T(t)v^2(t,T)dt \\
&= Z^T(t)v^2(t,T)dt - Z^T(t)v(t,T)dV(t)
\end{aligned}
$$

If we now make a Girsanov transformation from Q to Q^T with the kernel g(t) we get

$$
dZ^T(t) = Z^T(t)\left\{v^2(t,T) - g(t)v(t,T)\right\}dt - Z^T(t)v(t,T)dV^T(t)
$$

where V^T is a Q^T-Wiener process. We see that Z^T becomes a Q^T-martingale if $g(t) = v(t,T)$. This Girsanov kernel is the same at the one in the theorem above. Thus Q^T makes all prices martingales with $p(t, T)$ as numeraire.

Theorem 17.5. *If S is a process such as* $S(t)/B(t)$ *is a Q-martingale. Then, the process*

$$
Z^T(t) = \frac{S(t)}{p(t,T)}
$$

is a Q^T*-martingale on the interval* $[0, T]$.

Under Q^T, also

$$
\frac{\Pi(t,X)}{p(t,T)}
$$

is a Q^T-martingale. By using $p(T, T) = 1$ and $\Pi(T, X) = X$ we get

$$
\frac{\Pi(t,X)}{p(t,T)} = E^T\left[\frac{\Pi(T,X)}{p(T,T)} \mid \mathcal{F}_t\right] = E^T\left[X \mid \mathcal{F}_t\right]
$$

This is the **forward price** at time t for the contract X.

17.1 Forwards and Futures

Let us study a simple example where we need the forward measure, to value a forward or a future contract on underlying equity.

We suppose there exist a martingale measure Q. If we buy a T-contract today, at time t we know that the price is given by

$$\pi_t[X] = E^Q\left[X \cdot \exp\left\{ -\int_t^T r(s)ds \right\} | \mathcal{F}_t \right]$$

The cash flows are:

1.) At time t we pay the amount $\pi_t[X]$.
2.) At time T we receive the stochastic amount X.

A forward and a future contract are variants of the contract above, but they differ on how the cash flows are paid. We start with the simplest, the forward contract.

Definition 17.2. Let X be a contingent T-claim. With a **forward contract** on X contracted at time t, with the delivery at T and the **forward price** $\varphi(t, T, X)$ we mean the following construction:

(i) The holder of the forward contract receives at time T the stochastic amount X cash units.
(ii) The holder of the forward contract pays at time T the amount $\varphi(t, T, X)$ cash units.
(iii) The forward price $\varphi(t, T, X)$ is determined when we sign the contract at time t.
(iv) The forward price is determined so that the arbitrage-free price at the contract is equal zero when we sign the contract at time t.

The forward contract defined as above are traded OTC (Over-The-Counter) and not at exchanges. An important characteristic of the contract is the value of zero when the contract is signed at time t. Our problem is to find the mathematical price of the contract X of the construction above. This is quite obvious:

$$\begin{aligned}
0 = \pi_t[X - \varphi(t,T,X)] &= E^Q\left[[X - \varphi(t,T,X)] \cdot \exp\left\{ -\int_t^T r(s)ds \right\} | \mathcal{F}_t \right] \\
&= E^Q\left[X \cdot \exp\left\{ -\int_t^T r(s)ds \right\} | \mathcal{F}_t \right] - \varphi(t,T,X) \cdot E^Q\left[\exp\left\{ -\int_t^T r(s)ds \right\} | \mathcal{F}_t \right] \\
&= \pi_t[X] - \varphi(t,T,X)\, p(t,T)
\end{aligned}$$

where $p(t, T)$ is the price at time t of a zero coupon bond paying one cash unit at maturity T. We can summarize this and write down the price of the forward contract

$$\varphi(t,T,X) = \frac{\pi_t[X]}{p(t,T)} = \frac{1}{p(t,T)} \cdot E^Q\left[\exp\left\{-\int_t^T r(s)ds\right\} | \mathcal{F}_t\right]$$

We write this as

$$\varphi(t,T,X) = E^T\left[X \mid \mathcal{F}_t\right]$$

where E^T means integration with respect to the **forward measure** Q^T.

The reason that this contract is not traded at exchanges is credit risk, i.e., that the other party cannot fulfil the obligation. Therefore, another contract, which can be traded, standardized on exchanges, has been created. This is the future contract.

Definition 17.3. Let X be a contingent T-claim. With a **future contract** on X contracted at time t, with the **future price** $\Phi(t, T, X)$ we mean the following construction:

(i) For each time t there exist a price, $\Phi(t, T, X)$ called the **future price** for X with delivery at T.
(ii) At time T the holder of the contract pays $\Phi(t, T, X)$ and receive X cash units.
(iii) During each time interval $(t, t + dt]$ the holder receives $\Phi(t + dt, T, X)$ - $\Phi(t, T, X)$ cash units.
(iv) The market price at each time is equal to zero

In practical situations, the time dt is one bank day but can also be a week or a month. (iii) means that there are continuous cash flows between the buyer and the seller of the contract. In such construction, the credit risk is minimized to the change of the price during a period of dt. The buyer and the seller also have to hold a margin requirement on an account on the market place. This margin can be used to close the contract at time T.

We can notice the following about the future contract.

1.) $\Phi(T, T, X) = X$ so at time T there are no reason to deliver anything. This is also true in real situations.
2.) A future contract is not a contract where you have to deliver any security at T with a predefined price.
3.) There are no costs to enter or leave a future contract.
4.) The only part of the contract is the cash flows during the life time calculated as the price difference during a period dt in time.

We will now give a mathematical definition of the future contract.

Definition 17.4. A **future contract** on a contingent *T*-claim *X* is a security with an adapted price- and dividend process [π, Φ] given by the following conditions:

$$\Phi(t, T, X) = X, \quad P \text{ - almost true}$$
$$\pi_t = 0 \quad P \text{ - almost true}, \ \forall t \leq T$$

Theorem 17.9. *The price of a future contract is given by*

$$\Phi(t,T,X) = E^Q\left[X \mid \mathcal{F}_t\right]$$

If the short rate r and *X* are independent the price of the forward- and future contract coincide. I.e.,:

$$\Phi(t,T,X) = \varphi(t,T,X) = E^Q\left[X \mid \mathcal{F}_t\right]$$

17.2 A general option pricing formula

As we have seen above, that for a forward measure Q^T "takes care of the stochasticity" on the interval [0, *T*]. This can be seen above in the theorem that states the pricing formula:

$$\Pi(t,X) = p(t,T) \cdot E^T\left[X \mid \mathcal{F}_t\right]$$

We will now show how the calculations of prices of interest rate options can be simplified by using forward measures. But first we give the following lemma:

Lemma 17.10. *For a fixed T, then the forward rate process f(t, T) is a* Q^T*-martingale. Especially, for all t ≤ T we have:*

$$E^T\left[r(T) \mid \mathcal{F}_t\right] = f(t,T)$$

Proof: From a theorem above we have

$$
\begin{aligned}
E^T\left[r(T) \mid \mathcal{F}_t\right] &= \frac{1}{p(t,T)} E^Q\left[r(T)\exp\left\{-\int_t^T r(s)ds\right\} \mid \mathcal{F}_t\right] \\
&= -\frac{1}{p(t,T)} E^Q\left[\frac{\partial}{\partial T}\exp\left\{-\int_t^T r(s)ds\right\} \mid \mathcal{F}_t\right] \\
&= -\frac{1}{p(t,T)} \frac{\partial}{\partial T} E^Q\left[\exp\left\{-\int_t^T r(s)ds\right\} \mid \mathcal{F}_t\right] \\
&= -\frac{p_T(t,T)}{p(t,T)} = f(t,T)
\end{aligned}
$$

To simplify the calculations of options, we suppose that the short rate $r(t)$ under Q is given by

$$dr(t) = \left\{a(t) + b(t)r(t)\right\} dt + \sigma(t)dV(t)$$

Then we know

1. $r(t)$ is a normal distributed process.
2. The bond prices have the form of

$$p(t,T) = \exp\left\{A(t,T) - B(t,T)r(t)\right\}$$

3. The price of a European call option on a S-Bond, with maturity T and strike K is given by:

$$
\begin{aligned}
\Pi(t,X) &= E^Q\left[g\left\{r(T)\right\}\exp\left\{-\int_0^T r(s)ds\right\} \mid \mathcal{F}_t\right] \\
X = g(r) &= \max\left(\exp\left\{A(T,S) - B(T,S)r(T)\right\} - K, 0\right)
\end{aligned}
$$

With the theorem above, the price of this contract can at $t = 0$ be written as:

$$\Pi\left[0,X\right] = p(0,T)\cdot E^T\left[g(r(T))\right]$$

This is a much simpler formula than before, since:

- We do not have to calculate $p(0, T)$. This value is given on the market!
- The expectation value is a simple integral instead of an double integral.

To use the pricing formula, we have to find the distribution of $r(T)$ under Q^T. But this is a simple procedure.

Theorem 17.11. *Suppose that the dynamics of r under Q is given by*

$$dr(t) = \{a(t) + b(t)r(t)\}\,dt + \sigma(t)dV(t)$$

Then r(T) is normal distributed under Q^T with:

$$E^T\left[r(T)\right] = f(0,T)$$

$$Var^T\left[r(T)\right] = Var^Q\left[r(T)\right] = E^T\left[\left(\int_0^T \sigma(s)\cdot\exp\left\{\int_s^T b(\tau)d\tau\right\}dV(s)\right)^2\right]$$

$$= \int_0^T \sigma^2(s)e^{2H(T,s)}ds$$

where H is defined by

$$H(t,s) = \int_s^t b(\tau)d\tau$$

To calculate the variance of $r(T)$ under Q^T, we first integrate the process of r, giving

$$r(T) = e^{H(T,0)}r(0) + \int_0^T e^{H(T,s)}a(s)ds + \int_0^T e^{H(T,s)}\sigma(s)dV(s)$$

We know that

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dV(t)$$

have the kernel $g(t) = v(t, T)$. If we change the measure to Q^T with this Girsanov transformation and use the process for an affine term structure

$$dp(t,T) = r(t)p(t,T)dt - \sigma(t,r(t))B(t,T)p(t,T)dV(t)$$

we get

$$v(t,T) = -\sigma(t)B(t,T)$$

where $v(t, T)$ is deterministic. After the Girsanov transformation we have

$$r(T) = e^{H(T,0)}r(0) + \int_0^T e^{H(T,s)}\{a(s) + v(s,T)\}\,ds + \int_0^T e^{H(T,s)}\sigma(s)dW^T(s)$$

We can sum up this in the following theorem

Theorem 17.12. *Suppose that the dynamics of r under Q is given by*

$$dr(t) = \{a(t) + b(t)r(t)\}dt + \sigma(t)dV(t)$$

Then the price of a European call option is given by

$$\Pi(t,X) = p(t,T)\int_{-\infty}^{\infty} \max\left(\exp\{A(T,S) - B(T,S)z\} - K, 0\right)\varphi(z)dz$$

where φ is the density of a normal distribution with the expectation value

$$m = E^T\left[r(T) \mid \mathcal{F}_t\right] = f(t,T)$$

and variance

$$v = Var^T\left[r(T)\right] = \int_t^T e^{2H(T,s)}\sigma^2(s)ds$$

The price is given by

$$\Pi[t,X] = \frac{p(t,T)}{\sqrt{2\pi v}}\int_{-\infty}^{\infty} g(r(T))\exp\left\{-\frac{(r(T)-m)^2}{2v^2}\right\}dr$$

We remember from the end of lecture notes I that a general payoff for a European call option is given by:

$$C_T = \max(S(T) - K, 0) = (S(T) - K)I_{\{S(T)>K\}}$$

where $I_{\{S(T)>K\}}$ is an indicator function equal to 1, if $S(T) > K$ and 0 else. We use $S(t)$ as any underlying security. We then have the arbitrage free price as

$$\begin{aligned}\Pi(0,X) &= E^Q\left[\frac{1}{B(T)}(S(T) - K)I_{\{S(T)\geq K\}}\right] \\ &= E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}S(T)I_{\{S(T)\geq K\}}\right] - K\cdot E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}I_{\{S(T)\geq K\}}\right] \\ &= A - B\end{aligned}$$

The first expectation value is, if we use $S(t)$ as numeraire, $S(T)$ discounted to a present value $S(0)$

$$A = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}S(T)I_{\{S(T)\geq K\}}\right] = S(0)E^S\left[I_{\{S(T)\geq K\}}\right] = S(0)Q^S\left(S(T)\geq K\right)$$

The second expectation value is, if we use $p(t, T)$ as numeraire, the price of a discount bond with maturity T

$$B = K\cdot E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}I_{\{S(T)\geq K\}}\right] = K\cdot p(0,T)E^T\left[I_{\{S(T)\geq K\}}\right]$$
$$= K\cdot p(0,T)Q^T\left(S(T)\geq K\right)$$

Then we can write

$$\Pi(0,X) = S(0)Q^S\left(S(T)\geq K\right) - K\cdot p(0,T)Q^T\left(S(T)\geq K\right)$$

where Q^T denotes the T-forward measure and Q^S the martingale measure for the numeraire process $S(t)$.

In order to use this formula in real situation we have to be able to calculate the probabilities above. Before we do the calculation we will repeat some parts discussed in earlier sections. We know that $S(t)/B(t)$ is Q-martingale, where

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}$$

Therefore

$$d\left(\frac{S(t)}{B(t)}\right) = \frac{S(t)}{B(t)}\sigma(t)dW(t)\,.$$

The zero-coupon bond price is given by

$$p(t,T) = E^Q\left[\exp\left\{-\int_t^T r(u)du\right\}|\mathcal{F}_t\right] = E^Q\left[\frac{B(t)}{B(T)}|\mathcal{F}_t\right]$$

so

$$\frac{p(t,T)}{B(t)} = E^Q\left[\frac{1}{B(T)}|\mathcal{F}_t\right]$$

is also a martingale. The T-forward price $F(t, T)$ of S is the price set at time t for delivery of S at time T with payment at time T, The value of the forward contract at t is zero, so

$$0 = E^Q\left[\frac{B(t)}{B(T)}\{S(T) - F(t,T)\} \mid \mathcal{F}_t\right] = B(t)E^Q\left[\frac{S(T)}{B(T)} \mid \mathcal{F}_t\right] - F(t,T)E^Q\left[\frac{B(t)}{B(T)} \mid \mathcal{F}_t\right]$$
$$= B(t)\frac{S(t)}{B(t)} - F(t,T)p(t,T) = S(t) - F(t,T)p(t,T)$$

Therefore,

$$F(t,T) = \frac{S(t)}{p(t,T)}$$

Definition 17.5. Any asset in the model whose price is always strictly positive can be taken as *numeraire*. We can denominate all other assets in units of this numeraire.

Example 17.14 Money Market Account as numeraire. At time t, a stock S is worth $S(t)/B(t)$ units of money market and a T-bond is worth $p(t, T)/B(t)$ units of money market.

Example 17.15 Bond as numeraire. At time $t < T$, a stock S is worth $F(t, T)$ units of a T-maturity bond and the T-maturity bond is worth 1 unit.

Theorem 17.16. *Let N be a numeraire, i.e., the price process for some asset whose price is always positive. Then Q^N defined by*

$$Q^N(A) = \frac{1}{N(0)}\int_A \frac{N(T)}{B(T)}dQ, \qquad \forall A \in \mathcal{F}_T$$

is risk-neutral for N. Q^N is called the risk-neutral measure for the numeraire N. Note: Q and Q^N are equivalent, i.e., have the same probability zero set, and

$$Q(A) = N(0)\int_A \frac{B(T)}{N(T)}dQ^N, \qquad \forall A \in \mathcal{F}_T$$

Proof: Because N is the price process of some asset $N/B(t)$ is martingale under Q. Therefore

$$Q^N(\Omega) = \frac{1}{N(0)}\int_\Omega \frac{N(T)}{B(T)}dQ = \frac{1}{N(0)}E^Q\left[\frac{N(T)}{B(T)}\right] = \frac{1}{N(0)}\frac{N(0)}{B(0)} = 1$$

and we see that Q^N is a probability measure. Let Y be a traded asset price. Under Q, $Y/B(t)$ is a martingale. We must show that under Q^N, Y/N is a martingale. Using

$$E^Q[X] = \int_\Omega X dQ = \int_\Omega X \frac{dQ}{dP} dP = E^P\left[\frac{dQ}{dP} X\right]$$

$$E^{Q^N}\left[\frac{Y(T)}{N(T)} \mid \mathcal{F}_t\right] = \frac{B(t)}{N(t)} E^Q\left[\frac{N(T)}{B(T)} \frac{Y(T)}{N(T)} \mid \mathcal{F}_t\right] = \frac{B(t)}{N(t)} E^Q\left[\frac{Y(T)}{B(T)} \mid \mathcal{F}_t\right] =$$
$$\frac{B(t)}{N(t)} \frac{Y(t)}{B(t)} = \frac{Y(t)}{N(t)}$$

which is the martingale property for Y/N under the probability measure Q^N.

17.2.1 The Bond price as numeraire

Fix $T \in [0, T]$ and let $p(t, T)$ be the numeraire. The risk neutral measure for this numeraire is

$$Q^T(A) = \frac{1}{p(0,T)} \int_A \frac{p(T,T)}{B(T)} dQ = \frac{1}{p(0,T)} \int_A \frac{1}{B(T)} dQ, \qquad \forall A \in \mathcal{F}_T$$

Because the bond is not defined after the time T, we change measure only "up to time T", i.e., using

$$\frac{1}{p(0,T)} \frac{p(T,T)}{B(T)} \text{ and only for } A \in \mathcal{F}_T$$

Q^T is called the ***T*-forward measure**. Denominated in units of the T-maturity bond the value of the security S is

$$F(t,T) = \frac{S(t)}{p(t,T)}, \quad 0 \le t \le T$$

This is a martingale under Q^T and has the differential form:

$$dF(t,T) = \sigma_F(t,T) F(t,T) dW^T(t), \quad 0 \le t \le T$$

i.e., a differential without a drift term dt. The process $\{W^T; 0 \le t \le T\}$ is a Brownian motion under Q^T, and we may assume without loss of generality that $\sigma_F(t, T) \ge 0$.

Remark: The numeraire $p(t, T)$ is the price of the bond with maturity, T. Therefore, different forward neutral measures are not compatible against each other's. The value of $1 on maturity T cannot be equal to another measure.

17.2.2 The Stock price as numeraire

Let $S(t)$ be the numeraire. In terms of this numeraire, the stock price is identical to 1. The risk-neutral measure under this numeraire is

$$Q^S(A) = \frac{1}{S(0)} \int_A \frac{S(T)}{B(T)} dQ, \qquad \forall A \in \mathcal{F}_T$$

Denominated in shares of the stock, the value of the T-maturity bond is:

$$\frac{p(t,T)}{S(t)} = \frac{1}{F(t,T)}, \quad 0 \le t \le T$$

This is a martingale under Q^S and so has the differential form:

$$d\left(\frac{1}{F(t,T)}\right) = \gamma(t,T)\left(\frac{1}{F(t,T)}\right) dW^S(t)$$

Where $\{W^S; 0 \le t \le T\}$is a Brownian motion under Q^S, and we may assume without loss of generality that $\gamma(t, T) \ge 0$.

Theorem 17.17. *The volatility $\gamma(t, T)$ is equal to the volatility $\sigma_F(t, T)$. In other words we have*

$$d\left(\frac{1}{F(t,T)}\right) = \sigma_F(t,T)\left(\frac{1}{F(t,T)}\right) dW^S(t)$$

Proof: Let $g(x) = 1/x$, so $g'(x) = -1/x^2$, $g''(x) = 2/x^3$. Then

$$d\left(\frac{1}{F(t,T)}\right) = dg\big(F(t,T)\big) = g'\big(F(t,T)\big)dF(t,T) + \frac{1}{2}g''\big(F(t,T)\big)\big(dF(t,T)\big)^2$$
$$= -\frac{1}{F^2(t,T)}\sigma_F(t,T)F(t,T)dW^T(t) + \frac{1}{F^3(t,T)}\sigma^2{}_F(t,T)F^2(t,T)dt$$
$$= \frac{1}{F(t,T)}\left[-\sigma_F(t,T)dW^T(t) + \sigma^2{}_F(t,T)dt\right]$$
$$= \sigma_F(t,T)\frac{1}{F(t,T)}\left[-dW^T(t) + \sigma_F(t,T)dt\right]$$

Under Q^T, $-W^T$ is a Brownian motion. Under this measure $1/F(t, T)$ has volatility $\sigma_F(t, T)$ and mean rate of return $\sigma^2{}_F(t, T)$. The change of measure from Q^T to Q^S makes $1/F(t, T)$ a martingale, i.e., it changes the mean return to zero, but the change in measure does not affect the volatility. Therefore $\gamma(t, T)$ must be $\sigma_F(t, T)$ and W^S must be

$$W^S(t) = -W^T(t) + \int_0^t \sigma_F(u,T)du$$

We now turn back to the general pricing formula for a call option

$$\Pi(0,X) = S(0)Q^S\big(S(T) \ge K\big) - K \cdot p(0,T)Q^T\big(S(T) \ge K\big)$$

We assume that the process

$$Z(t) = \frac{S(t)}{p(t,T)}$$

has a stochastic differential of the form

$$dZ(t) = Z(t)m(t)dt + Z(t)\sigma(t)dW$$

We start to compute the second term:

$$Q^T\big(S(T) \ge K\big) = Q^T\left(\frac{S(T)}{p(T,T)} \ge K\right) = Q^T\big(Z(T) \ge K\big)$$

Since Z is an asset price, normalized by the price of a T-bond, it has zero drift under Q^T, so it is Q^T dynamics are given by

$$dZ(t) = Z(t)\sigma(t)dW^T$$

The solution to this is given by (use Itô's formula on ln(Z)):

$$
\begin{aligned}
d\ln\big(Z(t)\big) &= \frac{\partial}{\partial Z}\ln\big(Z(t)\big)dZ(t) + \frac{1}{2}\frac{\partial^2}{\partial Z^2}\ln\big(Z(t)\big)\big(dZ(t)\big)^2 \\
&= -\frac{1}{2}\sigma^2(t)dt + \sigma dW^T
\end{aligned}
$$

and we get

$$
\begin{aligned}
Z(T) &= \frac{S(0)}{p(0,T)}\exp\left\{\int_0^T \sigma(t)dW^T(t) - \frac{1}{2}\int_0^T \sigma^2(t)dt\right\} \\
&= \frac{S(0)}{p(0,T)}\exp\left\{\Sigma(T)\cdot W^T - \frac{1}{2}\Sigma^2(T)\right\}
\end{aligned}
$$

We know from stochastic calculus that the stochastic integral above has a normal distribution with mean zero and variance

$$
\Sigma^2(T) = \int_0^T \sigma^2(s)ds
$$

The entire exponent is thus normal distributed, and we can write the probability as

$$
\begin{aligned}
Q^T\big(Z(T)\geq K\big) &= Q^T\left(\frac{S(0)}{p(0,T)}\exp\left\{\Sigma\cdot W^T - \frac{1}{2}\Sigma^2(T)\right\}\geq K\right) \\
&= Q^T\left(\Sigma(T)\cdot W^T - \frac{1}{2}\Sigma^2(T)\geq \ln\left(\frac{K\cdot p(0,T)}{S(0)}\right)\right) \\
&= Q^T\left(\Sigma(T)\cdot W^T \geq \ln\left(\frac{K\cdot p(0,T)}{S(0)}\right) + \frac{1}{2}\Sigma^2(T)\right) \\
&= Q^T\left(-W^T \leq \frac{1}{\Sigma(T)}\left\{\ln\left(\frac{S(0)}{K\cdot p(0,T)}\right) - \frac{1}{2}\Sigma^2(T)\right\}\right) = N\big[d_2\big]
\end{aligned}
$$

where

$$
d_2 = \frac{\ln\left\{\dfrac{S(0)}{K\cdot p(0,T)}\right\} - \dfrac{1}{2}\Sigma^2(T)}{\Sigma(T)}
$$

The first probability in the option formula is a Q^S-probability we write

$$Q^S\left(S(T)\geq K\right)=Q^S\left(\frac{p(T,T)}{S(T)}\leq\frac{1}{K}\right)=Q^S\left(Y(T)\leq\frac{1}{K}\right)$$

where we defined $Y(t)$ as

$$Y(t)=\frac{p(t,T)}{S(t)}=\frac{1}{Z(t)}=\frac{p(0,T)}{S(0)}\exp\left\{\Sigma(T)\cdot W^S-\frac{1}{2}\Sigma^2(T)\right\}$$

Therefore

$$\begin{aligned}
Q^S\left(S(T)\geq K\right)&=Q^S\left(Y(T)\leq\frac{1}{K}\right)=Q^S\left(\frac{p(0,T)}{S(0)}\exp\left\{\Sigma(T)\cdot W^S-\frac{1}{2}\Sigma^2(T)\right\}\leq\frac{1}{K}\right)\\
&=Q^S\left(\Sigma(T)\cdot W^S-\frac{1}{2}\Sigma^2(T)\leq\ln\left(\frac{S(0)}{K\cdot p(0,T)}\right)\right)\\
&=Q^S\left(W^S\leq\frac{1}{\Sigma(T)}\left\{\ln\left(\frac{S(0)}{K\cdot p(0,T)}\right)+\frac{1}{2}\Sigma^2(T)\right\}\right)=N\left[d_1\right]
\end{aligned}$$

where

$$d_1=d_2+\Sigma(T)$$

To summarize, we have a general formula for call options

$$\Pi(0,X)=S(0)\cdot N[d_1]-K\cdot p(0,T)\cdot N[d_2]$$

where

$$d_2=\frac{\ln\left\{\dfrac{S(0)}{K\cdot p(0,T)}\right\}-\dfrac{1}{2}\Sigma^2(T)}{\Sigma(T)}$$

and

$$d_1=d_2+\Sigma(T)$$

Remark! If $r(t)$ is a constant, then $p(t,T)=e^{-rT}$ and we get the usual Black-Scholes formula.

17.2.3 The Hull-White Model

As a concrete application of the option pricing formula of the previous section, we will now consider the case of interest rate options in the simplified Hull-White model (the extended Vasicek model). To this end recall that in the Hull-White model the Q dynamics of $r(t)$ are given by

$$dr = \big(\theta(t) - ar\big)dt + \sigma dV(t)$$

We recall that we have an affine term structure

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)}$$

where $A(t, T)$ and $B(t, T)$ are deterministic functions, and where $B(t, T)$ is given by

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

The project is to price a European call option with date of maturity T_1 and strike price K, on an underlying bond with date of maturity T_2, where $T_1 < T_2$. In the notation of the general theory above, this means that $T = T_1$ and that $S(t) = p(t, T_2)$. We start by checking if the volatility, σ_Z, of the process

$$Z^T(t) = \frac{p(t,T_2)}{p(t,T_1)}$$

is deterministic. Inserting $p(t, T)$ into this gives

$$Z(t) = \exp\left\{A(t,T_2) - A(t,T_1) - \left[B(t,T_2) - B(t,T_1)\right]r(t)\right\}$$

Applying the Itô formula to this expression, we get the Q dynamics:

$$\begin{aligned} dZ(t) &= \frac{\partial}{\partial t}Z(t)dt + \frac{\partial}{\partial r}Z(t)dr + \frac{1}{2}\frac{\partial^2}{\partial r^2}Z(t)\sigma_Z^2 dt \\ &= Z(t)\{\ldots\}dt - Z(t)\cdot\{B(t,T_2) - B(t,T_1)\}\sigma dV \\ &= Z(t)\{\ldots\}dt + Z(t)\cdot\sigma_Z(t)dV \end{aligned}$$

i.e.,

$$\begin{aligned} \sigma_Z(t) &= -\sigma\{B(t,T_2) - B(t,T_1)\} = -\frac{\sigma}{a}\left\{1 - e^{-a(T_2-t)} - 1 + e^{-a(T_1-t)}\right\} \\ &= \frac{\sigma}{a}\left\{e^{-a(T_2-t)} - e^{-a(T_1-t)}\right\} = \frac{\sigma}{a}e^{at}\left\{e^{-aT_2} - e^{-aT_1}\right\} \end{aligned}$$

Thus σ_Z is in fact deterministic, so we may apply the option formula. We obtain the following result, which also holds for the Vasicek model.

The Hull-White bond option: In the Hull-White model the price, at $t = 0$, of a European call with strike price K, and time of maturity T_1, on a bond maturing at T_2 is given by the formula

$$\Pi(0,X) = p(0,T_2) \cdot N[d_1] - K \cdot p(0,T_1) \cdot N[d_2]$$

where

$$d_2 = \frac{\ln\left\{\dfrac{p(0,T_2)}{K \cdot p(0,T_1)}\right\} - \dfrac{1}{2}\Sigma^2}{\Sigma}$$

and

$$d_1 = d_2 + \Sigma$$

where

$$\begin{aligned}\Sigma^2 &= \int_0^T \sigma_z{}^2(s)ds = \frac{\sigma^2}{a^2}\left\{e^{-aT_2} - e^{-aT_1}\right\}^2 \int_0^{T_1} e^{2as} ds \\ &= \frac{\sigma^2}{2a^3}\left\{e^{-2aT_2} + e^{-2aT_1} - 2e^{-aT_1}e^{-aT_2}\right\}\left\{1 - e^{2aT_1}\right\} \\ &= \frac{\sigma^2}{2a^3}\left\{e^{-2a(T_2-T_1)} + 1 - 2e^{-a(T_2-T_1)}\right\}\left\{1 - e^{2aT_1}\right\}e^{-2aT_1} \\ &= \frac{\sigma^2}{2a^3}\left\{1 - e^{-2aT_1}\right\}\left\{1 - e^{-a(T_2-T_1)}\right\}^2\end{aligned}$$

We have used from above the formula

$$\Pi(0,X) = S(0) \cdot N[d_1] - K \cdot p(0,T) \cdot N[d_2]$$

and replaced the stock price $S(0)$ with the discount underlying bond $p(0, T_2)$ with maturity at T_2 and T with time to maturity for the bond option, T_1.

We end the discussion of the Hull-White model, by studying the pricing problem for a claim of the form

$$Z = \Phi\big(r(T)\big)$$

Using the T-bond as numeraire, we have by using the forward measure

$$\Pi\big(t,Z\big) = p(t,T)E_{t,r}^T\left[\Phi\big(r(T)\big)\right],$$

so we must find the distribution of $r(T)$ under Q^T, and to this we will use the volatility of the T-bond, and we obtain bond prices (under Q) as

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dV(t)$$

where the volatility $v(t, T)$ is given by

$$v(t,T) = -\sigma(t)B(t,T)$$

Thus, the Q^T-dynamics of the short rate are given by

$$dr = \left(\theta(t) - ar - \sigma^2 v(t,T)\right)dt + \sigma dV^T(t)$$

where V^T is a Q^T-Wiener process. We observe that, since $v(t, T)$ and $\theta(t)$ are deterministic, r is a Gaussian process, so the distribution of $r(T)$ is completely determined by its mean and variance under Q^T. Solving the linear SDE above gives us

$$r(T) = e^{-a(T-t)}r(t) + \int_t^T e^{-a(T-s)}\left[\theta(s) - \sigma^2 v(s,T)\right]ds + \sigma\int_t^T e^{-a(T-s)}dV^T(s)$$

We can now compute the conditional Q^T-variance of $r(T)$, $\sigma_r^2(t,T)$, as

$$\sigma_r^2(t,T) = \sigma^2\int_t^T e^{-2a(T-s)}ds = \frac{\sigma^2}{2a}\left\{1 - e^{-2a(T-t)}\right\}$$

Note that the Q^T-mean of $r(T)$, does not have to be computed at all, since we have

$$m_r(t,T) = E_{t,r}^T\left[r(T)\right] = f(t,T)$$

which can be observed directly from market data. Under Q^T, the conditional distribution of $r(T)$ is thus the normal distribution $N[f(t, T), \sigma_r(t, T)]$, and performing the integration we have the final result.

Given the assumptions above, the price of the claim

$$X = \Phi\left(r(T)\right)$$

is given by

$$\Pi\left(t,X\right) = p(t,T)\frac{1}{\sqrt{2\pi\sigma_r^2(t,T)}}\int_{-\infty}^{\infty}\Phi\left(z\right)\exp\left\{-\frac{\left(z - f(t,T)\right)^2}{2\sigma_r^2(t,T)}\right\}dz$$

17.2.4 The General Gaussian Model

In this section we extend our earlier results, by computing prices of bond options in a general Gaussian forward rate model. We specify the model (under Q) as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dV(t)$$

where V is a d-dimensional Q-Wiener process. We assume that the volatility vector function

$$\sigma(t,T) = \left[\sigma_1(t,T),...,\sigma_p(t,T)\right]$$

is a **deterministic** function of the variables t and T. Using the bond price dynamics under Q given by

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)v(t,T)dV(t)$$

where the volatility is given by

$$v(t,T) = -\int_t^T \sigma(t,s)ds$$

we consider a European call option, with expiration date T_0 and exercise price K, on an underlying bond with maturity T_1 (where of course $T_0 < T_1$). In order to compute the price of the bond, we use the pricing formula above, which means that we first have to find the volatility σ_{T_1,T_0} of the process

$$Z(t) = \frac{p(t,T_1)}{p(t,T_0)}$$

in easy calculation shows that in fact

$$\sigma_{T_1,T_0} = v(t,T_1) - v(t,T_0) = -\int_{T_0}^{T_1} \sigma(t,s)ds$$

This is clearly deterministic. We now have the following pricing formula for prices of Gaussian forward rates. The price, at $t = 0$, of the option

$$X = \max\left\{p(T_0,T_1) - K, 0\right\}$$

is given by

$$\Pi(0,X) = p(0,T_1)\cdot N[d_1] - K\cdot p(0,T_0)\cdot N[d_2]$$

where

$$d_2 = \frac{\ln\left\{\dfrac{p(0,T_1)}{K \cdot p(0,T_0)}\right\} - \dfrac{1}{2}\Sigma^2_{T_0,T_1}}{\sqrt{\Sigma^2_{T_0,T_1}}}$$

and

$$d_1 = d_2 + \sqrt{\Sigma^2_{T_0,T_1}}$$

and

$$\Sigma^2_{T_0,T_1} = \int_0^{T_0} \| \sigma_{T_0,T_1}(s) \|^2 \, ds$$