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Pricing of Bonds

14.1 Bond Pricing

As we have seen the price of a zero coupon bond at $t = 0$ and time to maturity T is given by

$$p(0,T) = E_{t,r}^{Q}\left[\exp\left\{-\int_{0}^{T} r(s)ds\right\}\right]$$

Therefore we can write the price of a coupon-bearing bond as

$$\begin{aligned} B(0,T) &= \sum_{n:t_n \geqslant 0}^{M} \frac{C \cdot N}{\omega} E_{t,r}^{Q}\left[\exp\left\{-\int_{0}^{t_n} r(s)ds\right\}\right] \\ &\quad + N \cdot E_{t,r}^{Q}\left[\exp\left\{-\int_{0}^{T} r(s)ds\right\}\right] \\ &= \frac{C \cdot N}{\omega} \sum_{n:t_n \geqslant 0}^{M} p(0,t_n) + N \cdot p(0,T) \end{aligned}$$

where N is the nominal amount, C the coupon rate, ω the coupon frequency, M the number of coupons and t_n the cash-flow dates. We use $p(0,t_n)$ as discount factors so the value of a bond on a particular date is completely determined by the discount curve at that date. We notice that at each time t_n, the bond price has a jump of size CN/ω.

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Thus, the value of the bond changes discontinuously. We can make the "price" continuous if we subtract the accrued interest rate AI. The usual convention is to let the coupon accrues linearly between the payouts. This accrued interest rate is the earned rate by the bondholder.

$$AI(t, t_n) = \frac{C \cdot N}{\omega} \frac{t - t_n}{t_{n+1} - t_n}$$

By definition, the **clean price** of a bond corresponds to the price at which the transaction takes place without including accrued interest. The **dirty price** is the price including accrued interest, that is, how much money trades hands (so to speak). Hence,

$$Dirty\ price = Clean\ price + AI\,(t,\ t_n)\,.$$

In an arbitrage-free economy, the dirty price should be equal to the theoretical value. In particular, the theoretical clean price can be expressed in terms of the term structure of interest rates as

$$B(0, T) = \frac{C \cdot N}{\omega} \sum_{n:t_n \geqslant 0}^{M} p(0, t_n) + N \cdot p(0, T) - AI(0, t_n)$$

The clean and dirty prices coincide on the coupon date after the coupon is paid (since $AI(t_n, t_n) = 0$. Bond quotes in the US Treasury, international and corporate markets are usually in terms of clean prices.

The yield of a bond (or *ytm*) is the effective constant interest rate that makes the bond price equal to the future cash flows discounted at this rate. The *ytm* is usually computed using the same frequency as the bond's interest payments (e.g. semi-annual), rather than the continuously compounded yield used for zeros.

Assume that the current date coincides with a coupon payment date, so that $t = t_m$. In this case, we define the *ytm* of the bond (after the coupon was paid) to be the value of Y such that

$$B(0, T) = \frac{C \cdot N}{\omega} \sum_{n=m+1}^{M} \left(\frac{1}{1 + Y/\omega}\right)^{n-m} + N \cdot \left(\frac{1}{1 + Y/\omega}\right)^{N-m}$$

If the current date does not coincide with a coupon date, we should take into account the fraction of year corresponding to the period between now and the next coupon date.

Accordingly, assume that $t_m < t < t_{m+1}$ and that f represents the ratio of the number of days in remaining until the next coupon date and the number of days in the coupon period, using the appropriate day count convention. (Hence, $0 < f < 1$). The bond yield Y is defined by the relation

$$B(0,T) = \frac{C \cdot N}{\omega} \sum_{n=m+1}^{M} \left(\frac{1}{1+Y/\omega}\right)^{f+n-m-1} + N \cdot \left(\frac{1}{1+Y/\omega}\right)^{f+N-m-1}$$

The previous equations define Y implicitly in terms of the bond value. It is easy to see that B is a decreasing function of Y. Moreover, B is convex in Y. To obtain the yield from the bond value, the equations must be solved numerically. Nevertheless, the yield of a bond is a well-defined function of its theoretical value B (the dirty price) and thus of the discount factors.

Notice that if $t = t_m$ we can use the summation formula for a geometric series to obtain

$$B(0,T) = \frac{C \cdot N}{\omega} \left(1 - \left(\frac{1}{1+Y/\omega}\right)^{N-m}\right) + N \cdot \left(\frac{1}{1+Y/\omega}\right)^{N-m}$$

This formula shows that if the yield is equal to the coupon rate, the value of the bond is equal to its face value. From this fact and the monotonicity of the price/yield relationship, we can derive some elementary relationships between price, yield and coupon.

If, immediately after a coupon payment, a bond trades at 100% of the principal, we say that the bond trades at par. In this case, its yield is exactly equal to the coupon rate. If the bond price is less than 100% of face value, we say that the bond trades at a discount. In this case, its yield is higher than the coupon rate. If the bond trades above 100% of face value, we say that bond trades at a premium. In this case, the yield is lower than the coupon rate.

In an arbitrage-free market, two bonds with same price and same cash-flow dates cannot have different coupons (otherwise, we can short the one with the smaller coupon and buy the one with the larger one). Similarly, two bonds with the same price and payment dates cannot have different yields. The notion of **par yield** – the yield of a par bond –is sometimes used to represent the term structure of interest rates implied by the bond market. In this case, one speaks of the par yield curve.

14.1.1 Duration

The price-yield relation gives rise to several quantities that are commonly used in bond risk-management. The first notion is that of duration (or average duration, or McCauley duration) which is defined as

$$D = \frac{1}{B}\left(\frac{C \cdot N}{\omega}\sum_{n:t_n \geqslant 0}^{M} t_n \cdot p(0,t_n) + N \cdot T \cdot p(0,T)\right)$$

$$= \frac{\frac{C \cdot N}{\omega}\sum_{n:t_n \geqslant 0}^{M} t_n \cdot p(0,t_n) + N \cdot T \cdot p(0,T)}{\frac{C \cdot N}{\omega}\sum_{n:t_n \geqslant 0}^{M} p(0,t_n) + N \cdot p(0,T)}$$

Thus, the duration represents a weighted average of the cash-flow dates, weighted by the cash flows measured in constant dollars. Mathematically, it is the "barycentre" of the cash-flow dates.

A closely related quantity is obtained by differentiating the bond price with respect to the *ytm*

$$\frac{\partial B}{\partial Y} = -\frac{C \cdot N}{\omega}\sum_{n=m+1}^{M}\left(\frac{f+n-m-1}{\omega}\right)\left(\frac{1}{1+Y/\omega}\right)^{f+n-m}$$
$$-N \cdot \left(\frac{f+N-m-1}{\omega}\right)\left(\frac{1}{1+Y/\omega}\right)^{f+N-m}$$

It follows from the definition of f, that $(f+n-m-1)$ represents the time between t and t_n measured in coupon periods ($1/\omega$ years). Therefore, the number $(f+n-m-1)/\omega$ represents the time interval between t and the n^{th} coupon date measured in years. We conclude that

$$\frac{1}{B}\frac{\partial B}{\partial Y} = -\frac{D}{1+Y/\omega}$$

Thus, the per cent sensitivity of the bond (dirty) price with respect to yield is of opposite sign and proportional to the average duration. The quantity

$$D_{\text{mod}} = \frac{D}{1+Y/\omega}$$

which represents the exact magnitude of the percentage change which is known as modified duration. These equations express the fact that the longer the duration, the greater the sensitivity of a bond to a change in yield, in percentage terms.

Clearly, a zero-coupon bond has duration equal to the time to maturity. The duration of a coupon-bearing bond trading at par (face value) immediately after the coupon date is

$$D = \frac{1}{\omega}\sum_{n=0}^{M-1}\frac{1}{(1+Y/\omega)^n} = \frac{1}{Y}\cdot(1+Y/\omega)\cdot\left(1-\frac{1}{(1+Y/\omega)^N}\right)$$

(The derivation of this formula is left as an exercise to the reader.) The formula shows that duration decreases with frequency. In fact, if the bond matures in T years and makes only a single payment, we have $N = 1, \omega = 1/T$. Substituting these values into the previous equation, we find $D = T$, the result for zeros. In the limit $\omega >> 1$, setting $N = \omega T$, we have $D = (1 - e^{-YT})/Y$.

The duration of a coupon-bearing bond is always smaller than the time-to-maturity, because far-away cash-flow dates are "discounted" more than nearby dates. We also get a formula for the modified duration of a par bond, which gives the price-yield sensitivity as

$$D_{\text{mod}} = \frac{1}{Y}\cdot\left(1-\frac{1}{(1+Y/\omega)^N}\right)$$

These formulas are useful for estimating the price-yield sensitivity of bonds. For example, if $N >> 1$ we can make the approximation $D_{mod} \approx 1/Y$. This approximation is exact for **perpetual** or **console bonds**, which are fixed income securities that pay a fixed coupon and have no redemption date. Because the maturity is infinite, the aforementioned formulas apply even if the console bond is not trading at par, by simply scaling the coupon. The modified duration of a console is exactly equal to $1/Y$. Moreover, it is easy to see that $Y = CN/B$.

Treasury bond prices are usually quoted in clean price or yield and bonds usually trade close to par (this is true for recently issued bonds). Historically, duration was introduced as a measure of the risk-exposure of a bond portfolio and hence as a hedging tool. The rationale for this is that if we assume that bond yields vary in the same direction and by the same amount, that is, if the yield curve shifts in parallel, we can measure the total exposure of a portfolio to a shift in the yield curve. In fact, a portfolio consisting of M bonds with n_1 dollars invested in bond, n_2 dollars invested in bond 2, etc., has, under the parallel shift assumption, a first-order variation with respect to yield of

$$\sum_j n_j \frac{dB_j}{B_j} = \left(\sum_j n_j D_{\text{mod}\,j} \right) dY$$

Thus, the sensitivity to a parallel shift in yields is equal to the dollar-weighted modified duration of the portfolio. A portfolio with vanishing dollar-weighted modified duration has no exposure to parallel shifts in the yield curve.

It has been recognized now for quite some time that duration-based hedging (under the explicit assumption of parallel shifts of the yield curve) is not precise enough to immunize a fixed income portfolio against interest rate risk. The reason is that yields of deferent maturities generally do not move together and by the same amount. Appropriate modelling of yield correlations is needed to produce efficient portfolio hedges and to correctly price fixed-income derivatives that are contingent on more than one yield. The modelling of yield correlations is an interesting subject.