13

Martingale Measures

13.1 Introduction to Martingale Measures

From now on, we will consider the filtrated probability space (Ω, $\mathcal{F}$, P, $\underline{\mathcal{F}}$) as given where W is a $\underline{\mathcal{F}}$-Wiener process on $[0, T]$. The interpretation is that we consider an economy on $[0, T]$ where all randomness is generated by W. The time horizon is needed to perform a number of Girsanov transformations in the interval $[0, T]$
We start with the following assumptions:

1. For each $T \geq 0$ there exist an adapted price process $p(t, T)$ for T-bonds.
2. There exists a local risk-free security with the price process B given by:

$$dB(t) = r(t)B(t)dtB(0) = 1$$

   where the short rate is given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

3. There exist a probability measure $Q \sim P$ such as each Z^T-process is a Q-martingale on $[0, T]$, where the discounted bond prices Z^T is defined as

$$Z^T(t) = \frac{p(t, T)}{B(t)}$$

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With the aforementioned assumptions, it easy to show that the bond prices have stochastic differentials. It is also possible to find the relations between the bond price and the short rate. First we notice that:

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}$$

Theorem 13.1.1. *With the previous assumption, we have for each fixed T*:

(i) *The bond prices for* $t \leq s \leq T$ *is given by*:

$$p(t,T) = E^Q\left[p(s,T)\exp\left\{-\int_t^s r(u)du\right\}|F_t\right]$$

Especial, with $s = T$, by

$$p(t,T) = E^Q\left[\exp\left\{-\int_t^T r(u)du\right\}|F_t\right]$$

(ii) *There exist adapted processes m(t, T) and ν(t, T) such as:*

$$dp(t,T) = m(t,T)p(t,T)dt + \nu(t,T)p(t,T)dW(t)$$

(iii) *The Q-dynamics of p(t, T) is given by:*

$$dp(t,T) = r(t)p(t,T)dt + \nu(t,T)p(t,T)dV(t)$$

where V is a Q-Wiener process.

(iv) *The Q-dynamics of the forward rates f(t, T) is given by:*

$$df(t,T) = \nu(t,T)\nu_T(t,T)dt - \nu_T(t,T)dV(t)$$

Proof (i): Since Z^T is a Q-martingale, we have:

$$\frac{p(t,T)}{B(t)} = Z^T(t) = E^Q\left[Z^T(s)\,|F_t\right] = E^Q\left[\frac{p(s,T)}{B(s)}\,|F_t\right]$$

so

$$p(t,T) = E^Q\left[p(s,T)\frac{B(t)}{B(s)}\,|F_t\right] = E^Q\left[p(s,T)\exp\left\{-\int_t^s r(u)du\right\}|F_t\right]$$

Proof (iii): To prove this we will use the reverse of the Girsanov theorem, which says that Q has arised from P via a Girsanov transformation:

$$dQ = L(T^*)dP \quad \text{on } \mathcal{F}_{T^*}$$

where

$$\begin{cases} dL(t) = \phi(t)L(t)dW(t) \\ L(0) = 1 \end{cases}$$

for some process $\phi(t)$. From Girsanov theorem we get

$$dW(t) = \phi(t)\boldsymbol{dt} + dV(t)$$

where V is a Q-Wiener process. By taking Itô on Z^T we get

$$\begin{aligned} dZ^T(t) &= \frac{\partial Z^T(t)}{\partial p(t,T)}dp + \frac{\partial Z^T(t)}{\partial B(t)}dB \\ &= \frac{1}{B(t)}\left\{p(t,T)m(t,T)\boldsymbol{dt} + p(t,T)v(t,T)dW(t)\right\} \\ &\quad - \frac{1}{B^2(t)}p(t,T)B(t)r(r)\boldsymbol{dt} \\ &= Z^T(t)\left\{m(t,T) - r(t)\right\}\boldsymbol{dt} + Z^T(t)v(t,T)dW(t) \\ &= Z^T(t)\left\{m(t,T) - r(t)\right\}\boldsymbol{dt} + Z^T(t)v(t,T)\left\{\phi(t)\boldsymbol{dt} + dV(t)\right\} \\ &= Z^T(t)\left\{m(t,T) - r(t) + v(t,T)\phi(t)\right\}\boldsymbol{dt} + Z^T(t)v(t,T)dV(t) \end{aligned}$$

With a choice of $\phi(t) = (r(t) - m(t,T))/v(t,T)$ we have the Q-dynamics

$$dZ^T(t) = v^T(t)Z^T(t)dV(t)$$

By definition, we have

$$p(t,T) = B(t)Z(t,T)$$

where $B(t)$ and $Z(t,T)$ have stochastic differentials under Q. Therefore we use Itô on the previous expression, and get (since the second order derivatives are zero)

$$\begin{aligned} dp(t,T) &= \frac{\partial p(t,T)}{\partial Z^T(t)} dZ^T(t) + \frac{\partial p(t,T)}{\partial B(t)} dB(t) \\ &= B(t)dZ^T(t) + Z^T(t)dB(t) \\ &= B(t)\nu^T(t)Z^T(t)dV(t) + Z^T(t)r(t)B(t)\boldsymbol{dt} \\ &= r(t)p(t,T)\boldsymbol{dt} + \nu^T(t)p(t,T)dV(t) \end{aligned}$$

This proves (iii). If we insert $dV(t)$ we get

$$\begin{aligned} dp(t,T) &= r(t)p(t,T)\boldsymbol{dt} + \nu^T(t)p(t,T)\left\{dW(t) - \phi(t)\boldsymbol{dt}\right\} \\ &= \left\{r(t) - \phi(t)\nu^T(t)\right\} p(t,T)\boldsymbol{dt} + \nu^T(t)p(t,T)dW(t) \end{aligned}$$

Therefore, under P, we have $m^P(t,T) = r(t) - \phi(t)\nu(t,T)$.

To **prove (iv)** we use (iii) which say that under $Q: m(t,T) = r(t)$. This gives

$$\frac{\partial m(t,T)}{\partial T} = \frac{\partial r(t)}{\partial T} = 0$$

The relation between dp^T and df^T was given via

$$\begin{aligned} dp(t,T) &= m(t,T)p(t,T)\boldsymbol{dt} + \nu(t,T)p(t,T)dW(t) \\ df(t,T) &= \alpha(t,T)\boldsymbol{dt} + \sigma(t,T)dW(t) \end{aligned}$$

where

$$\begin{aligned} \alpha(t,T) &= \nu_T(t,T)\nu(t,T) - m_T(t,T) \\ \sigma(t,T) &= -\nu_T(t,T) \end{aligned}$$

which gives

$$df(t,T) = \left\{\nu_T(t,T)\nu(t,T) - m_T(t,T)\right\}\boldsymbol{dt} - \nu_T(t,T)dW(t)$$

$\Rightarrow$

$$df(t,T) = \nu(t,T)\nu_T(t,T)\boldsymbol{dt} - \nu_T(t,T)dV(t)$$

Here we have also used that $m(t,T) = r(t) \Rightarrow \phi(t) = 0 \Rightarrow dW(t) = dV(t)$ Therefore we have proved (iv).

The results (i) – (iii) are pretty expected. But (iv) is a little bit of surprise since this shows that under Q there must exist a relationship between the drift and diffusion for the forward rates. In other word: The dynamic of the forward rates under Q is uniquely determined by the diffusion coefficient. This will be essential in a later section where we will study the Heath-Jarrow-Morton framework.

Since we have

$$dF^T(t) = \{r(t) - \lambda(t)\sigma_T(t)\}\, F^T(t)dt + \sigma_T(t)F^T(t)dW(t)$$

where $F^T = p(t,T)$, we have that $\phi(t) = -\lambda(t)$.

Before we will show that the model is free of arbitrage, we will give some **definitions**.

Definition 13.1.0.2. A *portfolio strategy* is a finite adapted process h:

$$h(t) = \left\{h^0(t), h(t,T_1), \ldots, h(t,T_n)\right\}$$

where by definition $h(t,T_k) = 0$ for $t > T_k$. Furthermore:

$$\int_0^{T^*} \left|h^0(t)\right| dt < \infty \qquad P \text{ a.s.}$$

$$E^Q\left[\int_0^{T^*} \{h(t,T_k)Z(t,T_k)\}^2\, dt\right] < \infty \quad k = 1, \ldots n$$

Definition 13.1.0.3. Given a portfolio strategy h, the *value process* $V(h)$ is defined by:

$$V_t(h) = h^0(t)B(t) + \sum_{k=1}^{n} h(t,T_k)p(t,T_k)$$

Definition 13.1.0.4. A portfolio h is said to be *self-financing* if

$$dV_t(h) = h^0(t)dB(t) + \sum_{k=1}^{n} h(t,T_k)dp(t,T_k)$$

Definition 13.1.0.5. The class of self-financing portfolios is denoted by $\boldsymbol{H}$. A *contingent claim* is a stochastic variable X such as

$$X \text{ is } \mathcal{F}_{T^*}\text{–measurable.}$$
$$E^Q[X^2] < \infty$$

The class of contingent claims is denoted by $\boldsymbol{K}$. With $\boldsymbol{K}^+$ we refer to those $X \in \boldsymbol{K}$ such as

$$P(X \geq 0) = 1, \quad \text{and} \quad P(X > 0) > 0$$

Definition 13.1.0.6. A contingent claim is said to be *reachable* on $[0, T]$ if there exist a self-financing strategy h such as

$$V^T(h) = X, \qquad P\text{–a.s.}$$

Definition 13.1.0.7. A self-financing strategy h is said to be an *arbitrage strategy* if there exist a time T such as

$$V^T(h) \in \boldsymbol{K}^+, \text{ and } V^0(h) = 0$$

With the earlier definitions, the number of T-bonds in the portfolio at the time t is given by $h(t, T)$ and $h^0(t)$ the number of the risk-free security. Due to our definition, we have at $t = 0$ decide the number of possible bonds in our portfolio. The rollover strategy discussed in an earlier section is not allowed in the previous portfolio strategy. But we will still consider the short rate r in terms of the rollover. As before we can move to the discounted Z-economy to show that the model is free of arbitrage.

Lemma 13.1.8. *For a given portfolio strategy h, we define $V^Z(h)$ as*

$$V_t^Z(h) = h^0(t) + \sum_{k=1}^{n} h(t, T_k)Z(t, T_k)$$

Then

$$V_t(h) = B(t)V_t^Z(h)$$

The strategy is self-financed if and only if

$$dV_t^Z(h) = \sum_{k=1}^{n} h(t, T_k)dZ(t, T_k)$$

If h is self-financed, then V^Z becomes a Q-martingale.

Theorem 13.1.9. *With the assumptions* 1, 2 *and* 3, *the model is free of arbitrage.*

Theorem 13.1.10. *With the assumptions* 1, 2 *and* 3, *v(t, T) ≠ 0 for all (t, T) with* $0 \leq t \leq T$. *Then*:

(i) *The money market is complete, that is, each contingent claim is reachable via an self-financing portfolio. More precise, if X is a contingent claim* $X \in \mathcal{F}^T$, *then it is possible to replicate X with a portfolio of T-bonds only and the risk-free security.*

(ii) *For X as given earlier, the arbitrage free price is given by:*

$$\pi_t[X] = E^Q\left[X \cdot \exp\left\{-\int_t^T r(u)du\right\} |\mathcal{F}_t\right]$$

Proof: If X is reachable via a portfolio h we know that V^T is a Q-martingale. Then:

$$V_t^Z(h) = E^Q\left[V_T^Z(h)\,|\mathcal{F}_t\right] = E^Q\left[\frac{V_T(h)}{B(T)}\,|\mathcal{F}_t\right] = E^Q\left[\frac{X}{B(T)}\,|\mathcal{F}_t\right]$$

so

$$\begin{aligned} V_t(h) &= B(t)V_t^Z(h) = B(t)E^Q\left[\frac{X}{B(T)}\,|\mathcal{F}_t\right] \\ &= E^Q\left[X\frac{B(t)}{B(T)}\,|\mathcal{F}_t\right] = E^Q\left[X \cdot \exp\left\{-\int_t^T r(u)du\right\} |\mathcal{F}_t\right] \end{aligned}$$

Theorem 13.1.11. *Suppose r given on* $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$. *Then, there exist an infinite number of arbitrage-free term structures for this r. More precisely, for each Girsanov kernel* ϕ, *the bond prices can be defined by:*

$$p(t,T) = E^Q\left[\exp\left\{-\int_t^T r(u)du\right\} |\mathcal{F}_t\right]$$

where Q is defined by

$$df(t,T) = v(t,T)v_T(t,T)dt - v_T(t,T)dV(t)$$

and

$$\begin{cases} dL(t) = \phi(t)L(t)dW(t) \\ L(0) = 1 \end{cases}$$

The term structure is then free of arbitrage. Furthermore, the Girsanov kernel ϕ is related to the market price of risk, λ such as $\lambda = -\phi$. That is, λ is the Girsanov kernel for the transformation from Q to P.

Theorem 13.1.12. *Suppose the short rate r on the martingale measure Q solves the SDE*

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dV(t)$$

and let X be a contingent claim: $X = \Phi[r(T)]$. Then, the price of X on Q is

$$\pi_t[X] = F[t, r(t)]$$

where F is a solution to the PDE

$$\begin{cases} \dfrac{\partial F(t,x)}{\partial t} + \mu(t,x)\dfrac{\partial F(t,x)}{\partial x} + \dfrac{1}{2}\sigma^2(t,x)\dfrac{\partial^2 F(t,x)}{\partial x^2} - xF(t,x) = 0 \\ F(T,x) = \Phi(x) \end{cases}$$

To calculate arbitrage prices via a PDE, r has to be a Markov process on the martingale measure Q. r is a Markov process from the suggestions:

(i) We supposed that r on P was a solution to a SDE.

(ii) We supposed that the market price of risk was a function of time and interest rate.