12
Term Structures

12.1 The Term Structure of Interest Rates

We will now consider the problem where we will model price processes on an arbitrage-free market of zero coupon bonds. On this market we will model the short rate, $r(t)$ under the real probability measure P. The process of the short rate will be given as

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

The only possibility to invest Capital is via roll-over:

$$dB(t) = r(t)B(t)dt$$

Therefore we can say that, to make rollover on the bank is equivalent by holding the security for which the price process is given by dB.
We now make the following assumptions:

- The interest rate $r(t)$ is a stochastic process.
- There exists only one security B with the aforementioned dynamic.
- All other securities are considered as derivatives of this $(r(t))$.

This means that we will consider a bond as an interest derivative where the value of the bond depends on the expectation of the future development of the short interest rate $r(t)$. We want to use arbitrage arguments to say something about the bond prices. It will be more difficult to analyse the market of interest rate derivatives than the simple Black-Scholes market we have been studying so far.

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Remark! According to the **Meta Theorem,** we have only one known security, B and one random source. Therefore **the market of interest rates is free of arbitrage but not complete.**

When we were studying the Black-Scholes market we did also know the price of the underlying stock. Therefore we can guess that, as soon we **know** the price of at least one bond, then we can price all other bonds relative this one, and the known security B. This is also true according to the Meta theorem.
We now suppose that we have one T-bond with a price given at t as:

$$p(t,T) = F(r(t), t, T) = F^T(r(t), t)$$

where F is a real function with tree real variables. Sometimes we will consider T as a parameter. We ask ourselves about the properties of the function F so that the Capital market is free of arbitrage. As we can see, we have a simple boundary condition

$$F(r, T, T) = 1 \text{ for all } r.$$

We will now create portfolios of bonds with different time to maturity T. Therefore we need the dynamics of the function F. By using the Itô formula we get

$$\begin{aligned} dF^T &= \frac{\partial F^T}{\partial t} dt + \frac{\partial F^T}{\partial r} dr + \frac{1}{2} \frac{\partial^2 F^T}{\partial r^2} (dr)^2 \\ &= F_t^T dt + F_r^T \{\mu dt + \sigma dW\} + \frac{1}{2}\sigma^2 F_{rr}^T dt \\ &= \left\{ F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T \right\} dt + \sigma F_r^T dW = F^T \alpha_T dt + F^T \sigma_T dW \end{aligned}$$

where

$$\begin{cases} \alpha_T = \dfrac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T}, \\ \sigma_T = \dfrac{\sigma F_r^T}{F^T} \end{cases}$$

Let us now fix two times S and T and study self-financing portfolios based on bonds with maturities S and T. As usually the Capital of such a portfolio will be described by a value process given by

$$V = h^T F^T + h^S F^S$$

To become self-financing, we must have

$$dV = h^T \cdot dF^T + h^S \cdot dF^S$$

If we use relative portfolios we have

$$dV = V\left\{u^T \frac{dF^T}{F^T} + u^S \frac{dF^S}{F^S}\right\} = V\left\{u^T \alpha_T + u^S \alpha_S\right\} dt + V\left\{u^T \sigma_T + u^S \sigma_S\right\} dW$$

where

$$u^T + u^S = 1$$

Since we only have one random source (one Wiener process), we can make the following choice to eliminate the last bracket in dV in the previous equation, that is, we have

$$\begin{cases} u^T \sigma_T + u^S \sigma_S = 0 \\ u^T + u^S = 1 \end{cases}$$

then, after some algebra we get

$$dV = V\left\{\frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S}\right\} dt$$

In a market free of arbitrage, we must have

$$dV = rVdt$$

That is,

$$\frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} = r$$

With some algebra

$$\begin{aligned} &\alpha_S \sigma_T - \alpha_T \sigma_S = r(\sigma_T - \sigma_S) \\ &\alpha_S \sigma_T - r\sigma_T = \alpha_T \sigma_S - r\sigma_S \\ &\sigma_T(\alpha_S - r) = \sigma_S(\alpha_T - r) \end{aligned}$$

We can also write this as

$$(*) \qquad \frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t, r)$$

so, we do not have any dependencies between S and T. The function $\lambda(t)$ is called **the market price of risk**. We can see this from

$$dF^T = F^T\alpha_T dt + F^T\sigma_T dW = F^T\{r + \lambda\sigma_T\}\,dt + F^T\sigma_T dW$$

So, $\lambda(t, r)$ is a risk premium per unit of volatility. We measure the risk in volatility. If we insert the definitions of α_T and σ_T in $(*)$ we get

$$\alpha_T(t) - r(t) - \lambda(t, r)\sigma_T(t) = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T} - r(t) - \lambda(t, r)\frac{\sigma F_r^T}{F^T} = 0$$

$$\Rightarrow$$

$$F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - F^T r(t) - \lambda(t, r)\sigma F_r^T = 0$$

So, we get the partial differential equation

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + \{\mu(t, r) - \lambda(t, r)\sigma(t)\}\dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 F^T}{\partial r^2} - r(t)F^T = 0 \\ F(r, T, T) = 1 \end{cases}$$

This equation is, in the literature called the **equation of the term structure**[1] or the term structure equation (TSE). Remark! This is a Black-Scholes equation where we replaced μ with $\mu - \lambda\sigma$. However, this PDE is more complex since λ is a unknown function: $\lambda = \lambda(r(t), t)$.

12.1.1 Yield- and Price Volatility

In fixed income it is very important to distinguish between yield-volatility and price volatility. In the process for the short rate we have the volatility for the yield, and the process for the bond prices we have the volatility for the prices.

As we saw earlier, if we did start with a process for the rate as

$$dr(t) = \mu dt + \sigma_r dW(t)$$

[1] A better name is the Bond Pricing PDE, since the “Term” T is a fixed time and not a variable.

and let the zero coupon price be given by $F = F(t, T, r)$ and use Itô on F we get

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial r}dr + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}(dr)^2 = F_t dt + F_r\{\mu dt + \sigma_r dW\} + \frac{1}{2}\sigma_r^2 F_{rr} dt$$
$$= \left\{F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr}\right\} dt + \sigma_r F_r dW = F\alpha dt + F\sigma_p dW$$

We saw before that the relationship between these volatilities follows

$$\sigma_p = \frac{\sigma_r F_r}{F} \equiv \frac{\sigma_r}{p(t,T)}\frac{\partial p(t,T)}{\partial r} \equiv \sigma_r \cdot D_{\text{mod}}$$

where D_{mod} is the modified duration. If the interest short rate is log-normal distributed, that is,

$$dr(t) = \mu \cdot r \cdot dt + \sigma_r \cdot r \cdot dW(t)$$

we would be given the following relationship

$$\sigma_p = r \cdot \frac{\sigma_r F_r}{F} \equiv r \cdot \frac{\sigma_r}{p(t,T)}\frac{\partial p(t,T)}{\partial r} \equiv r \cdot \sigma_r \cdot D_{\text{mod}}$$

This formula is often used by traders. The true relationship seems to be somewhere in between these results and depends on the time to maturity for the zero coupon bond.

12.1.1.1 Measuring Historical Yield Volatility

We know that volatility is measured in terms of the standard deviation or variance. To find the historical yield volatility we start with the daily data on yields. This can be from bonds quoted in *ytm* or other data of similar kind. We denote an interest rate on day t as y_t. It is important to choose the right number of days T that the volatility measure is going to be based on. Different number of observations would result in a different volatility estimate. Typically, portfolio managers with the longer investment horizon use a greater number of observations when calculating the volatility of interest rates.

We start by computing the daily relative yield change, X_t, assuming continuous compounding

$$X_t = 100 \cdot \ln \frac{y_t}{y_{t-1}}$$

Then we compute the daily standard deviation of yields

$$\sigma_{day} = \sqrt{Var\,(X_t)} = \sqrt{\frac{\sum_t (X_t - \bar{X})^2}{T-1}}$$

where $\bar{X}$ is the mean value of X_t and T the number of measurements (maturity). Some market practitioners argue that in forecasting volatility the expected value or mean that should be used in the formula for variance is zero.

Next, we annualize the standard deviation of yields

$$\sigma_{annual} = \sigma_{day} \cdot \sqrt{D}$$

where D is the number of trading-days per year. Analysts can use different number of days in a year in this step, but the usual practice is to exclude holidays (~10 days a year) from calculations, so that the number of trading days is $5 \times 52 - 10$ holidays = 250 trading days.

How do we interpret yield volatility? Let us assume, for example, that the annualized interest rate volatility of a 5-year note is 10%. Further, let us assume that currently the yield on this note is 2%. The standard deviation of interest rates on this bond would then equal 10% x 2% = 0.2% (20 basis points). Having calculated this standard deviation, an analyst would be able to estimate the confidence interval for interest rates. For example, a 95% confidence interval can be estimated as $2\% + -1.96 \times 0.2\%$.

12.1.1.2 Historical Versus Implied Yield Volatility

The procedure for calculation of yield volatility described previously is based on historical yield data. Another approach is to derive volatility from the valuation of options, in which case it is called the implied volatility. We assume that the options are currently trading near their fair value and derive the yield volatility estimate from the option pricing model. Swaptions are usually quoted on the market as Black volatility.

There are several problems with using implied volatility.

- It is based on the assumption that the option pricing model is correct.
- Models make the simplifying assumption that volatility is constant.
- Options may not be fairly priced by the market, which results in a misleading estimate of implied volatility.

12.1.1.3 Forecasting Yield Volatility

There are three different approaches to forecasting the volatility of interest rates

- Yield volatility forecast equals the variance based on the last T days with the mean yield change assumed to be zero.
- Similar to the first approach, but the formula gives more weight to the more recent interest rate changes. More specifically, observations further in the past should be given less weight.
- Statistical models of time series, such as autoregressive conditional heteroskedasticity (ARCH) model, may also be employed to forecast yield volatility. The ARCH model can incorporate trends in volatility, such as the observation that periods of low volatility are followed by periods of high volatility and vice versa.

12.1.2 The Market Price of Risk

The TSE contains references to the functions $\mu - \lambda\sigma$ and σ. The former is the coefficient of the first-order derivative with respect to the spot rate, and the latter appears in the coefficient of the diffusive, second-order derivative. The four terms in the equation represent, in order as written, time decay, drift, diffusion and discounting. The equation is similar to the backward equation for a probability density function, except for the final discounting term. As such we can interpret the solution of the bond pricing equation as the expected present value of all cash flows. As with equity options, this expectation is not with respect to the real random variable, but instead with respect to the risk-neutral variable. There is this difference because the drift term in the equation is not the drift of the real spot rate μ, but the drift of another rate, called the risk-neutral spot rate. This rate has a drift of

$\mu - \lambda\sigma$. When pricing interest rate derivatives (including bonds of finite maturity) it is important to model, and price, using the risk-neutral rate. This rate satisfies

$$dr = (\mu - \lambda\sigma)dt + \sigma dW.$$

We need the new market-price-of-risk term because our modelled variable, r, is not traded. If we set λ to zero then any results we find are applicable to the real world. If, for example, if we want to find the distribution of the spot interest rate at some time in the future then we would solve a Fokker-Planck equation with the real, and not the risk-neutral, drift. Because, we cannot observe the function λ, except possibly via the whole yield curve.

12.1.3 Solutions to the TSE

The solution to a SDE as the TSE can be represented in an integral form in terms of the underlying stochastic process.

$$F(t,s) = E_t\left[\exp\left(-\int_t^s r(u)du - \frac{1}{2}\int_t^s \lambda^2(u, r(u))du - \int_t^s \lambda(u, r(u))dW(u)\right)\right]$$

To prove this, we define

$$V(s) = \exp\left(-\int_t^s r(u)du - \frac{1}{2}\int_t^s \lambda^2(u, r(u))du - \int_t^s \lambda(u, r(u))dW(u)\right)$$

Now, let us differentiate the process $F(t,s)V(t)$. Let $f = FV$ and then $df = d(FV)$

$$\begin{aligned} d(FV) &= \frac{\partial(FV)}{\partial F}dF + \frac{\partial(FV)}{\partial V}dV + \frac{\partial^2(FV)}{\partial V \partial F}dVdF = VdF + FdV + dFdV \\ &= V\left(F\left\{r + \lambda\sigma_T\right\}dt + F\sigma_T dW\right) + FdV \\ &\quad + \left(F\left\{r + \lambda\sigma_T\right\}dt + F\sigma_T dW\right)dV \end{aligned}$$

with

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial W}dW + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}(dW)^2$$
$$= V\left(-r - \frac{1}{2}\lambda^2\right)dt - V\lambda dW + \frac{1}{2}\lambda^2 V dt$$
$$= -rVdt - \lambda V dW$$

we get

$$d(FV) = FV\left(\{r + \lambda\sigma_T\}\,dt + \sigma_T dW\right) + FdV$$
$$- FV\left(\{r + \lambda\sigma_T\}\,dt + \sigma_T dW\right)(rdt + \lambda dW)$$
$$= FV\left(\{r + \lambda\sigma_T\}\,dt + \sigma_T dW\right) - FV\,(rdt + \lambda dW) - FV\sigma_T\lambda dt$$
$$= FV\,(\sigma_T - \lambda)\,dW$$

By integrating from t to s and taking expectation value[2] yields (the term dW will be zero)

$$E_t\left[F(s,s)V(s) - F(t,s)V(t)\right] = 0$$

Since $F(s,s) = 1$, $V(t) = 1$ $E_t\left[V(s) - F(t,s)\right] = 0$ and $F(t,s) = E[V(s)]$.
TSE can also be solved using standard numerical methods such as finite difference methods. In several cases, analytical solutions also exist for the discount functions and European options. The only distinction between instruments is the boundary conditions. The equation is linear, so the superposition principle holds, that is, when all instruments in a portfolio fulfil the equation, the value of the portfolio also fulfils the equation.

In the first term structure models, the form of the functions $\mu(r,t)$ and $\sigma(r,t)$ was specified, containing several parameters that had to be estimated from historical data, or implied from market prices. Also, the market risk price parameter $\lambda(r,t)$ was specified as a single number.

This is a preference-dependent parameter, that is, it may be different from trader to trader. This means that there is no guarantee that the model generates a term structure that agrees with the observed term structure. This is a serious limitation for pricing option elements, since

[2] With expectation value, we always refer to the conditional expectation value, the conditional information known up to a certain time. This time is usual today, since we donot know anything about the future!

any small mispricing of the underlying instruments could mean major mispricing of the options.

Some authors still proposed trading strategies where all model parameters were derived from statistical analysis of historical prices. The strategy tries to find mispriced bonds, where the mispricing is likely to disappear.

12.1.4 Relative Pricing

To overcome these serious limitations of the early pricing models, Ho & Lee (1987) took a new approach. Their model assumed that the whole term structure followed a random evolution.

The model is still one-dimensional, since there is only one stochastic influence. They developed their model in a discrete time, binomial framework. A few years ago many said that this model had a serious disadvantage, since the Ho & Lee model are based on a stochastic evolution of the term structure that generate negative interest rates with positive probability. Now days we know that negative interest rates can occur.

Two papers, Jamshidan (1990) and Hull & White (1990) describe how to adjust the market price of the risk parameter $\lambda(r, t)$ in order to obtain consistency between model prices and the observed term structure of interest rates.

Their approach makes it possible to use the flexibility of the equilibrium models to specify stochastic processes together with the adjusted risk parameters, which generates a term structure consistent with what is observed.

The models resulted in a PDE, which can be solved by standard numerical techniques such as finite differences. Some of the models considered use normally distributed interest rates. In many cases, these models have analytical solutions for the discount function and for European options.

As usual, we can use the **Feynman-Kač representation** on the TSE:

$$(*) \quad F(r, t, T) = E_{t,r}^{Q}\left[\exp\left\{-\int_{t}^{T} r(s)ds\right\}\right]$$

where the Q-dynamics of r is given by

$$\begin{cases} dr(s) = \{\mu(s) - \lambda(s)\sigma(s)\}\, ds + \sigma(s)dV(s) \\ r(t) = r \end{cases}$$

<u>Conclusions</u>: The equation (∗) is incredible simple. If we write it as

$$F(r,t,T) = E^{Q}_{t,r}\left[\exp\left\{-\int_t^T r(s)ds\right\} \times 1\right]$$

we see that the bond price is given as the expectation value of \$1 (£1, 1 Kr. . .) paid at maturity T, discounted to a per cent value. The expectation value is calculated, not with respect to the objective probability measure P, but using the risk adjusted martingale measure Q that depends on $\lambda(t)$. That is, we get a new martingale measure for each $\lambda(t)$, so that the measure Q is not unique. This is because the model is not complete. In Black-Scholes world on the other hand, the martingale measure is unique and the model is complete. The interest market is **not** complete because we only have one given security.

The reason of having different martingale measures for different market prices of risk, $\lambda(t)$ is because of the reason that we can have many different markets, free of arbitrage and consistent with the short rate r. The bond prices on each market will depend on the liquidity and the traders will to enter risky positions. When we have a given market price of one bond, we know the market price of risk. Then we also know the prices of all other bonds.

The bond prices are therefore determined, partly of the P-dynamics of the short interest rate r and partly by the market. A general contingent claim $X = \Phi(r(t))$ is priced as

$$\Pi\,(t, X) = F\,(t, r\,(t)\,, T)$$

where

$$F(t,r,T) = E^{Q}_{t,r}\left[\exp\left\{-\int_t^T r(s)ds\right\} \Phi(r(T))\right]$$