11

Stochastic Processes

11.1 Pricing Theory

Modern pricing models generally use one of two powerful approaches; *equilibrium pricing* or *relative pricing*. In an equilibrium framework, certain market characteristics, such as a price risk, are estimated and the model can be used to predict prices for securities in the market. There is no guarantee that the model will price any security at its observed market price. In the relative pricing framework, some observed market prices are used as a starting point, and then other securities are priced relative these.

We will now start to consider the particular problems that appear when we try to apply arbitrage theory to the bond market. The primary objects of investigation are zero coupon bonds, also known as pure discount bonds, with various maturities. All payments are assumed to be made in a fixed currency (e.g. US dollars). Previously the short interest rates have been considered to be deterministic. In reality the interest rates are stochastic. This makes the theory of interest rate difficult and interesting.

We will begin with some definitions and then discuss the stochastic processes concerning the theory of interest rates.

Definition 11.1.0.1. A *zero coupon bond* with maturity date T, also called a T-bond, is a contract which guarantees the holder 1 (dollar, sterling, kronor ...) to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t, T)$ or $p^T(t)$.

The convention that the payment at the time of maturity, known as the **principal value**, **face value** or **nominal amount**, equals one,

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is made for computational convenience. Coupon bonds, which give the owner a payment stream during the interval $[0, T]$ are treated subsequently. These instruments have the common property that they provide the owner with a deterministic cash flow, and for this reason they are also known as **fixed income instruments**. The graph of $p(t, T)$ is called the **term structure of bond prices** at time t.

We assume the following:

- There exists a fix income market of T-bonds for all $T > 0$.
- $p(t, t) = 1$ for all times t.
- For a given $t, p(t, T)$ is differentiable with respect to T.
- At the bond market, $p(t, T)$ there exist an infinite number of securities.

We also define the derivative of the bond price $p(t, T)$ with respect to T as

$$p_T(t, T) = \frac{\partial p(t, T)}{\partial T}$$

A typical problem

We want to write a contract at time t that gives a deterministic interest rate in the interval $[S, T]$. We do this as:

1. At time t we sell one S-bond. This will give us $p(t, S)$ dollars.
2. We use this income to buy exactly $p(t, S)/p(t, T)T$-bonds. Thus our net
3. investment at time t equals zero.
4. At time S the S-bond matures, so we are obliged to pay out one dollar.
5. At time T the T-bonds mature at one dollar a piece, so we will receive the amount $p(t, S)/p(t, T)$ dollars.
6. The net effect of all this is that, based on a contract at t, an investment of one dollar at time S has yielded $p(t, S)/p(t, T)$ dollars at time T.
7. Thus, at time t, we have made a contract guaranteeing a risk-less rate of interest over the future interval $[S, T]$. Such an interest rate is called a **forward rate**.

11.1.1 Interest Rates

We will calculate relevant interest rates on the construction as shown earlier. We will use the **simple forward rate *L*** and the **continuous forward rate *R*** that solves:

$$1 + L\,(T - S) = \frac{p(t, S)}{p(t, T)}$$

and

$$e^{R(T-S)} = \frac{e^{-RS}}{e^{-RT}} = \frac{p(t, S)}{p(t, T)}$$

Definition 11.1.1.2. The *simple forward rate* for the period $[S, T]$ contracted at time t is defined by

$$L(t, S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)\,p(t, T)}$$

Definition 11.1.1.3. The *simple spot rate* for $[S, T]$ is defined by:

$$L(S, T) = -\frac{p(S, T) - p(S, S)}{(T - S)\,p(S, T)} = -\frac{p(S, T) - 1}{(T - S)\,p(S, T)}$$

Definition 11.1.1.4. For $t \leqslant S \leqslant T$ we define the *continuously compounded forward rate* for $[S, T]$ contracted at time t as

$$R(t, S, T) = -\frac{\ln\,[p(t, T)] - \ln\,[p(t, S)]}{T - S}$$

Definition 11.1.1.5. We define the *continuously compounded spot rate* for the period $[S, T]$ as

$$R(S, T) = -\frac{\ln\,[p(S, T)] - \ln\,[p(S, S)]}{T - S} = -\frac{\ln\,[p(S, T)]}{T - S}$$

Definition 11.1.1.6. Especial we define the *forward rate* for the period $[t, T]$ as

$$R(t, T) = R(t, t, T) = -\frac{\ln[p(t, T)]}{T - t}$$

Definition 11.1.1.7. The *instantaneous forward rate* with maturity at T contracted at t is defined as

$$f^T(t) = f(t, T) = \lim_{S \to T} R(t, S, T) = -\frac{\partial \left[\ln p(t, T)\right]}{\partial T}$$

giving

$$p(t, T) = \exp\left\{-\int_t^T f(t, u)du\right\}$$

This is equivalent as, at time contracted at time t as agree to pay \$1 at time T and then *receive* $e^{f(t,T)} \cdot \Delta T$. In terms of FRA, this is at time t agree to pay \$1 at time T_0 and then at time T receive

$$\exp\left\{\int_{T_0}^T f(t, u)du\right\}$$

or to pay

$$\exp\left\{-\int_{T_0}^T f(t, u)du\right\}$$

at time T_0 and then receive \$1 at time T.

Definition 11.1.1.8. The *instantaneous short rate* at time t is then defined by

$$r(t) = f(t, t)$$

We then have

$$p(t, T) = \exp\left\{-R(t, T) \cdot (T - t)\right\}$$

Before, we defined the **money account** by the process

$$\begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1 \end{cases}$$

giving

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}$$

Lemma 11.1.9. *The following holds for $t \leqslant s \leqslant T$:*

$$p(t,T) = p(t,s)\exp\left\{-\int_s^T f(t,u)du\right\} = \exp\left\{-\int_t^T f(t,u)du\right\}$$

When we study the interest rate market we have to start with something we know and depending on what choice we make, calculate what is unknown. Therefore we formulate the following questions:

1. If we let the dynamic of the short rate be given. Which bond prices $p(t,T)$ is consistent with the choice of r? Will the bond prices be uniquely given by r? Will these be free of arbitrage?
2. Which internal conditions do the bond prices $\{p^T; T \geqslant 0\}$ have to satisfy to have an arbitrage free money market?
3. Which internal conditions do the family of forward rates $\{f^T; T \geqslant 0\}$ have to satisfy to have an arbitrage free money market?
4. What can we say about the prices of different derivatives on an arbitrage-free money market?

To summarize what we have defined before, if we plot the interest rates, they form the term structure of interest rates or yield curve. We can represent the yield curve in three different but equivalent ways.

1. The first representation is by the prices of pure discount bonds (sometimes called zero-coupon bonds) that give the holder a single unit cash flow (e.g. one dollar) at maturity with no intermediate cash flows. We defined previously the function $p(t,T)$ to be the price, at time t, of a discount bond which matures at time T, with $t \leqslant T, (p(T,T) = 1)$. Remark! This is equivalent to the discount function defined earlier.
2. We can also represent the term structure by associating the continuously compounded spot rate $R(t,T)$ (sometimes called par yield)

with the pure discount bond price $p(t, T)$:

$$p(t, T) = e^{-R(t,T)(T-t)}$$

Inverting this equation we obtain

$$R(t, T) = -\frac{\ln[p(t, T)]}{T - t}$$

3. The third formulation is in terms of the forward rate curve, $f(t, T)$. This function represents at time t, the instantaneously maturing interest rate at time T and is derived from the discount bond function by applying the following transformation:

$$f(t, T) = -\frac{\partial \left[\ln p(t, T)\right]}{\partial T}$$

Combining the aforementioned equations, we can write the price of a pure discount bond as the final cash flow discounted by the instantaneous forward rates

$$p(t, T) = \exp\left\{-\int_t^T f(t, u)du\right\}$$

and the spot rate as the continuous average of forward rates:

$$R(t, T) = \frac{1}{T - t}\left(\int_t^T f(t, u)du\right)$$

For each of these rates, or prices, we associate a volatility. The function that describes these volatilities we call the term structure of interest rate volatilities. In terms of spot rates, a typical volatility structure exhibits short-term interest rates that are more volatile than longer-term interest rates an empirical feature of most markets. The effect is illustrated in Fig. 11.1.

The discount function is related to the bond prices as $D(T) = p(0, T)$, which is the value of \$1 paid at time T.

Fig. 11.1

11.1.2 Stochastic Processes for Interest Rates

From now on, we will think of a Wiener process W on a filtrated probability space $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ to generate the uncertainty. It will then be naturally to specify objects via Itô equations. $\underline{\mathcal{F}}$ is the natural filtration generated by the Wiener process and F the σ-algebra containing all the information on the sample space Ω.

We want to consider the stochastic processes for the short rate, the forward rate and the bond prices as follows

$$dr(t) = \mu(t)dt + \sigma(t)dW(t),$$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \text{ and}$$

$$dp(t,T) = m(t,T)p(t,T)dt + v(t,T)p(t,T)dW(t)$$

Here, μ and σ are adapted processes, defined for all times $t \geqslant 0$. For each fixed $T, m(t,T), v(t,T), \alpha(t,T)$ and $\sigma(t,T)$ are adapted processes for $0 \leqslant t \leqslant T$. We will also suppose that all the previous processes are continuous in t and two times differentiable. Further, we suppose that $v(T,T) = 0$ for all T. This seems to be OK since $p(T,T) = 1$ by definition.

We then have three choices. We can start with

- the dynamics of the short rate dr,
- the dynamics of the forward rates df^T or by
- the dynamics of the bond prices dp^T

We are now ready to study how these are related to each other.

11.1.2.1 The Relation from Bond Prices (dp^T) to Forward Rates (df^T)

We will start with the dynamics of the bond prices to see how this process is related to the process of the forward rates.

Therefore we start with

$$dp(t,T) = m(t,T)p(t,T)dt + v(t,T)p(t,T)dW(t)$$

If we integrate this process we get

$$p(t,T) = p(0,T) + \int_0^t p(u,T)m(u,T)du + \int_0^t p(u,T)v(u,T)dW(u)$$

Now, we take the derivative of this, and believe we can take the derivatives inside both of the integrals.

$$p_T(t,T) = p_T(0,T) + \int_0^t \{p_T(u,T)m(u,T) + p(u,T)m_T(u,T)\}\, du$$
$$+ \int_0^t \{p_T(u,T)v(u,T) + p(u,T)v_T(u,T)\}\, dW(u)$$

We then see that the stochastic differential of $p_T(t,T)$ is given by:

$$dp_T^T(t) = \left\{p_T^T(t)m^T(t) + p^T(t)m_T^T(t)\right\} dt + \left\{p_T^T(t)v^T(t) + p^T(t)v_T^T(t)\right\} dW(t)$$

We now use the definition of the instantaneous forward rate with maturity at T:

$$f^T(t) = -\frac{\partial \left[\ln p(t,T)\right]}{\partial T} \equiv -\frac{p_T(t,T)}{p(t,T)}$$

Set $p^T = p$ and so on, and use the Itô formula on $f = f^T$, we then get

$$df = \frac{\partial f}{\partial p_T} dp_T + \frac{\partial f}{\partial p} dp + \frac{1}{2}\frac{\partial^2 f}{\partial p_T^2}(dp_T)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial p^2}(dp)^2 + \frac{\partial^2 f}{\partial p \partial p_T} dp dp_T$$

The derivatives are given as

$$\frac{\partial f}{\partial p_T} = -\frac{1}{p}, \quad \frac{\partial^2 f}{\partial p_T^2} = 0, \quad \frac{\partial f}{\partial p} = \frac{p_T}{p^2}, \quad \frac{\partial^2 f}{\partial p^2} = -2\frac{p_T}{p^3} \quad \text{and} \quad \frac{\partial^2 f}{\partial p \partial p_T} = \frac{1}{p^2}$$

That is,

$$\begin{aligned} df &= -\frac{1}{p} dp_T + \frac{p_T}{p^2} dp - \frac{1}{2} 2 \frac{p_T}{p^3}(dp)^2 + \frac{1}{p^2} dp dp_T \\ &= -\frac{1}{p} dp_T + \frac{p_T}{p^2} dp - v^2 p^2 \frac{p_T}{p^3} dt + \frac{1}{p^2}\left\{p_T p v^2 + p^2 v_T v\right\} dt \\ &= v_T v dt - \frac{1}{p} dp_T + \frac{p_T}{p^2} dp \end{aligned}$$

where we have used

$$(dp)^2 = v^2 p^2 dt.$$

Since we just calculated dp_T and we know dp, we can just multiply the two expressions. To the lowest order we get:

$$dp_T dp = vp\left\{p_T v + p v_T\right\} dt = \left\{p_T p v^2 + p^2 v_T v\right\} dt$$

By putting these into the expression of df we find

$$\begin{aligned} df &= v_T v dt - \frac{1}{p}\left\{\left\{p_T m + p m_T\right\} dt + \left\{p_T v + p v_T\right\} dW\right\} + \frac{p_T}{p^2}\left\{mpdt + vpdW\right\} \\ &= \left\{v_T v - \frac{p_T}{p} m - m_T + \frac{p_T}{p} m\right\} dt + \left\{-\frac{p_T}{p} v - v_T + \frac{p_T}{p} v\right\} dW \\ &= \left\{v_T v - m_T\right\} dt - v_T dW = \alpha dt + \sigma dW \end{aligned}$$

where

$$\alpha(t,T) = v_T(t,T) v(t,T) - m_T(t,T)$$

and

$$\sigma(t,T) = -v_T(t,T)$$

To summarize with a known stochastic process for the bond prices:

$$dp(t,T) = m(t,T)p(t,T)dt + v(t,T)p(t,T)dW(t)$$

We can find the stochastic process for the forward prices $df(t,T)$:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)rdW(t)$$

where

$$\alpha(t,T) = v_T(t,T)v(t,T) - m_T(t,T)$$
$$\sigma(t,T) = -v_T(t,T)$$

11.1.2.2 The Relation from Forward Rates (df^T) to Short Rates (*dr*)

We will now start with the dynamics of the forward rates to see how this process is related to the process of the short rates.

Therefore we start with the stochastic process

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

If we integrate this process we get

$$f(t,T) = f(u,T) + \int_u^t \alpha(s,T)ds + \int_u^t \sigma(s,T)dW(s)$$

By using the definition $r(t) = f(t,t)$ and set $T = t$ and $u = 0$ we get:

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW(s)$$

But, remember that this is not a standard form of a stochastic differential since the processes depends on the integration limits. To overcome this difficulty, we write

$$\alpha(s,t) = \alpha(s,s) + \int_s^t \alpha_T(s,u)du$$

and

$$\sigma(s,t) = \sigma(s,s) + \int_s^t \sigma_T(s,u)du$$

To see this, imagine the integral

$$\int_s^t \alpha_T(s,u)du = \alpha(s,t) - \alpha(s,s)$$

Put these expressions into the integral for $r(t)$

$$r(t) = r(0) + \int_0^t f_T(0,s)ds + \int_0^t \alpha(s,s)ds$$
$$+ \int_0^t \int_s^t \alpha_T(s,u)duds + \int_0^t \sigma(s,s)dW(s) + \int_0^t \int_s^t \sigma_T(s,u)dudW(s)$$

Change the order of integration

$$r(t) = r(0) + \int_0^t f_T(0,s)ds + \int_0^t \alpha(s,s)ds + \int_0^t \int_0^u \alpha_T(s,u)dsdu +$$
$$+ \int_0^t \sigma(s,s)dW(s) + \int_0^t \int_0^u \sigma_T(s,u)dW(s)du$$

We can illustrate the change in the order of integration in Fig. 11.2. This explains the change in the integration limits. Before we change the order, u goes from 0 to t. Then s starts at u on the line $u = s$. In the next graph we have changed the order and then, when s goes from 0 to t, u does the same (i.e. from 0 to t).

At last, we use the process of the short rate

$$dr(t) = \mu(t)dt + \sigma(t)dW(t)$$

and integrate to find

$$r(t) = r(0) + \int_0^t \mu(u)du + \int_0^t \sigma(u)dW(u)$$

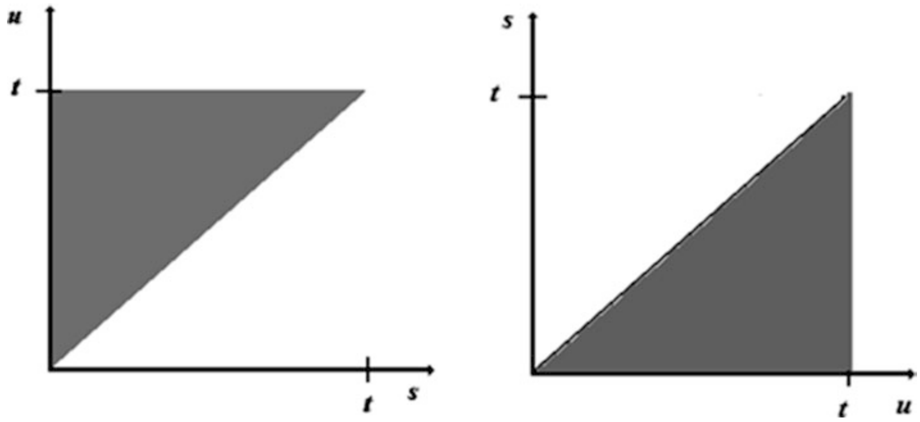

Fig. 11.2

Comparing with the previous expression we see that

$$\begin{cases} \mu(t) = f_T(0,t) + \alpha(t,t) + \int_0^t \alpha_T(s,t)ds + \int_0^t \sigma_T(s,t)dW(s) \\ \sigma(t) = \sigma(t,t) \end{cases}$$

If we use:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s)$$

$$\Rightarrow$$

$$f_T(t,T) = f_T(0,T) + \int_0^t \alpha_T(s,T)ds + \int_0^t \sigma_T(s,T)dW(s)$$

this can be simplified to

$$\begin{cases} \mu(t) = f_T(t,t) + \alpha(t,t) \\ \sigma(t) = \sigma(t,t) \end{cases}$$

11.1.2.3 The Relation From Forward Rates (df^T) to Bond Prices (dp^T)

We will start with the dynamics of the forward rates to see how this process is related to the process of the bond prices.

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

If we use the definition of the instantaneous forward rate with maturity at T

$$f^T(t) = -\frac{\partial\left[\ln p(t,T)\right]}{\partial T}$$

and write the bond prices as

$$p(t, T) = \exp\{Z(t, T)\}$$

where

$$Z(t,T) = -\int_t^T f(t,s)ds$$

Compare with the definition of the aforementioned bond price! We also know that

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)du + \int_0^t \sigma(u,s)dW(u)$$

We start by calculating $dZ(t,T)$

$$\begin{aligned} dZ(t,T) &= f(t,t)dt - \int_t^T df(t,u)du = r(t)dt - \int_t^T [\alpha(t,u)dt + \sigma(t,u)dW(t)]\,du \\ &= r(t)dt - \int_t^T \alpha(t,u)dudt - \int_t^T \sigma(t,u)dudW(t) \\ &= \left\{ r(t) - \int_t^T \alpha(t,u)du \right\} dt - \left\{ \int_t^T \sigma(t,u)du \right\} dW(t) \end{aligned}$$

where we have used

$$\frac{dZ(t,T)}{dt} = -\frac{\partial}{\partial t}\int_t^T f(\tau,u)du\,|_{\tau=t} - \int_t^T \frac{\partial}{\partial t} f(t,u)du = f(t,t) - \int_t^T df(t,u)$$

and

$$df(t,T) = \frac{\partial f(t,T)}{\partial t}dt + \frac{\partial f(t,T)}{\partial T}dT = \frac{\partial f(t,T)}{\partial t}dt$$

since we are studying one family of forward rates with maturity T. By using the Itô formula on $p(t,T) = \exp\{Z(t,T)\}$ we get:

$$\begin{aligned}
dp(t,T) &= \frac{\partial p}{\partial Z}dZ(t,T) + \frac{1}{2}\frac{\partial^2 p}{\partial Z^2}(dZ(t,T))^2 \\
&= p(t,T)dZ(t,T) + \frac{1}{2}p(t,T)(dZ(t,T))^2 \\
&= p(t,T)\left\{\left[r(t) - \int_t^T \alpha(t,u)du\right] + \frac{1}{2}\left(\int_t^T \sigma(t,u)du\right)^2\right\}dt \\
&\quad - p(t,T)\left\{\int_t^T \sigma(t,u)du\right\}dW(t)
\end{aligned}$$

This can be rewritten as

$$dp(t,T) = p(t,T)\{r(t) + b(t,T)\}dt + p(t,T)a(t,T)dW(t)$$

where we can identify $a(t,T)$ and $b(t,T)$

$$\begin{cases} a(t,T) = -\int_t^T \sigma(t,u)du \\ b(t,T) = -\int_t^T \alpha(t,u)du + \frac{1}{2}a(t,T)^2 \end{cases}$$

At last we have:

$$\begin{cases} m(t,T) = r(t) + b(t,T) \\ \nu(t,T) = a(t,T) \end{cases}$$

To summarize, we have that

- The forward rate $R(t,S,T)$ gives the average yield in the interval $[S,T]$ contracted at time t.
- The forward rate $f(t,T)$ gives the instantaneous yield at T contracted at time t.

- The short rate $r(t)$ gives the instantaneous yield of a T–bond contracted at time t. This is the yield of a portfolio strategy where we at each time t invest all of our Capital in the bond with immediately expiration.

The last strategy is called a **rollover-strategy** and its value process is given by:

$$dV(t) = V(t)u(t)\frac{dp(t,t)}{p(t,t)}$$

where $u(t)$ is the Capital at time t invested in the bond p^t. Per definition we have $u(t) = 1$ for all t and:

$$\frac{dp(t,t)}{p(t,t)} = \{r(t) + b(t,t)\}\, dt + a(t,t)dW(t)$$

But, since $a(t,t) = b(t,t) = 0$ we get

$$dV(t) = r(t)V(t)dt$$

The possibility to make a rollover-strategy on the bond market implies the existence of a local risk-free security with the stochastic yield given by $r(t)$