10

Option-Adjusted Spread

10.1 The OAS Model

A common method to value bonds, zero bonds and promissory loans with embedded options (that is, callable and putable instruments) is the use of option-adjusted spread (OAS). This model will use a spread on a benchmark curve to calculate bond prices for risky bonds, due to embedded options and since they are so called corporate bonds.

The model we will use is based on a Black-Derman-Toy (BDT) (see next) interest rate binomial tree approach and adjusts for the cost of the embedded option and the difference between model price and market price due to other risks, for example credit and liquidity risks.

The BDT model is a single-factor short-rate model matching the observed term structure of forward rate volatilities, as well as the term structure of the interest rate. A binomial tree is constructed for the short rate in such a way that the tree automatically returns the observed yield function and the volatility of different yields. The model is described by a SDE where the rates are log-normally distributed. Therefore, the interest rates cannot be negative.

To adjust the theoretical price on the binomial tree to the actual price, a spread (called option-adjusted spread since the context of OAS started with trying to correct for miss-pricing in option embedded securities) is added to all short rates on the binomial tree **such that the new model price after adding this spread makes the model price equal the market price** (this is the defining purpose of OAS).

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The value of OAS is that it enables investors to directly compare fixed income instruments, which have similar characteristics, but traded at significantly different yields because of embedded options.

The OAS model has three dependent variables:

- Option-adjusted spread
- Underlying price
- Volatility

The model can calculate the following model specific risk measures (except for the risk measures discussed earlier):

- Effective duration
- Effective modified duration
- Effective convexity
- Option-adjusted spread

10.1.1 Some Definitions

As we will see, the bullet bond can be used to find the "value" of the embedded option. For example, a callable bond of the option value is given by the price difference between the bullet bond and the callable bond.

There are six steps associated with the OAS analysis. The following assumption is that the method is being applied to a callable bond:

1. A binomial tree is built on dates where we have cash flows. Also create nodes in the tree where the instrument is callable or putable. Therefore extra nodes are added at the beginning and/or the ends of call periods (if they not coincide with cash flows).
2. Build a binomial tree using these rates, with equal probabilities (= $^1/_2$).
3. Calibrate the tree to market data by adjusting the nodes until the tree can replicate any cash flow as the discount function given by the benchmark yield curve.
4. Calibrate the model by adding the same number of basis points (the spread factor) to all rates in the tree until the model replicate the actual market price (if this price is known) of the callable bond. The result is the bond's OAS.

5. Apply the same OAS to value a bullet bond with terms identical to the callable/putable bond.
6. Take the difference between the value obtained for the callable bond and the value obtained for the bullet bond. This difference is the value of the embedded option.

The model creates nodes in the binomial tree on the following events:

1. Cash flows
2. Single call or put events
3. Start or end of call or put periods

No intermediate nodes are created, since there are no dynamical changes between the nodes. (For better accuracy in the interval where the bond can be called (or putted) back we can build intermediate nodes in other (all) parts of the tree.)

10.1.2 Building the Binomial Tree

The stochastic process for the short rate in the BDT model is given by

$$dr = a(t) \cdot r \cdot dt + \sigma(t) \cdot r \cdot dz$$

where $z(t)$ is a Brownian motion. In some literature this SDE is written as:

$$d\ln(r) = \{\theta(t) + \rho(t)\ln(r)\}\,dt + \sigma(t)dz$$

where $\theta(t)$ will be shown to be the drift of the short-term rate and $\rho(t)$ the mean reversing term to an equilibrium short-term rate that depends on the interest rate local volatility as follows

$$\rho(t) = \frac{d}{dt}\ln[\sigma(t)] = \frac{\dot{\sigma}(t)}{\sigma(t)}.$$

That is,

$$d\ln(r) = \left\{\theta(t) + \frac{\dot{\sigma}(t)}{\sigma(t)}\ln(r)\right\} dt + \sigma(t)dz$$

Since the volatility is time dependent, there are two independent functions of time, $\theta(t)$ and $\sigma(t)$, chosen so that the model fits the term structure of spot interest rates and the structure of the spot rate volatilities.

Jamshidan (1991) shows that the level of the short rate at time t in the BDT model is given by

$$r(t) = U(t) \exp\{\sigma(t)z(t)\}$$

where $U(t)$ is the median of the lognormal distribution of r at time t, $\sigma(t)$ the level of the short rate volatility and $z(t)$ the level of the Brownian motion, a normal distributed Wiener process that Captures the randomness of future changes in the short-term rate. One of the main advantages of the BDT model is that it is a lognormal model that is able to Capture a realistic term structure of the interest rate volatilities. To accomplish this feature, the short-term rate volatility is allowed to vary over time, and the drift in interest rate movements depends on the level of rates. Due to the property of Brownian motions, we have

$$z(t) = \varepsilon \cdot \sqrt{t}$$

where the values

$$\varepsilon = \begin{cases} +1 & \text{or} \\ -1 & \end{cases}$$

is used to build the tree. From the previous discussion, a fixed spacing, Z_i between the nodes in the binomial tree is defined as ($\varepsilon_{max} - \varepsilon_{min} = 2$):

$$Z_i = e^{2\sigma_i \cdot \sqrt{t_i - t_{i-1}}} \tag{10.1}$$

where σ_i is the volatility at time t. The risk-neutral probabilities of the binomial branches of this model are assumed equal to $1/2$. (It by no means implies that the actual probability for an interest rate increase or decrease is equal to $1/2$.) The tree uses the short-rate annual volatility, σ, of the benchmark rates which should be given in the Black-Scholes framework. The process can be illustrated using the following four short rates (all expressed with semi-annual compounding):

$$f_1 = 6.000\,\%$$
$$f_2 = 7.200\,\%$$
$$f_3 = 8.150\,\%$$
$$f_4 = 8.836\,\%$$

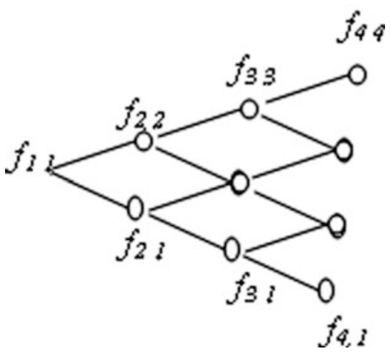

Fig. 10.1

Assume for simplicity that annual volatility of the short rates is constant, and given by 15%. When the tree is built, the volatility spread factors, Z_i are kept constant and the tree is built with the following relation between the nodes:

$$f_{i,j} = Z_i^{j-1} \cdot f_{i,1} \tag{10.2}$$

where $f_{1,1} = f_1$. This results in the tree in Fig. 10.1
where the rates is given by

$$\begin{cases} f_{2,2} = Z_2 \cdot f_{2,1} \\ \frac{1}{2} f_{2,1} + \frac{1}{2} f_{2,2} = f_2 \end{cases} \Rightarrow \quad f_{2,1} = \frac{2 \cdot f_2}{1 + Z_2} \quad \Rightarrow \quad f_{2,2}$$

$$\begin{cases} f_{3,3} = Z_3^2 \cdot f_{3,1} \\ f_{3,2} = Z_3 \cdot f_{3,1} \\ \frac{1}{4} f_{3,1} + \frac{1}{2} f_{3,2} + \frac{1}{4} f_{3,3} = f_3 \end{cases} \Rightarrow f_{3,1} = \frac{4 \cdot f_3}{1 + 2 \cdot Z_3 + Z_3^2} \Rightarrow f_{3,2} \Rightarrow f_{3,3}$$

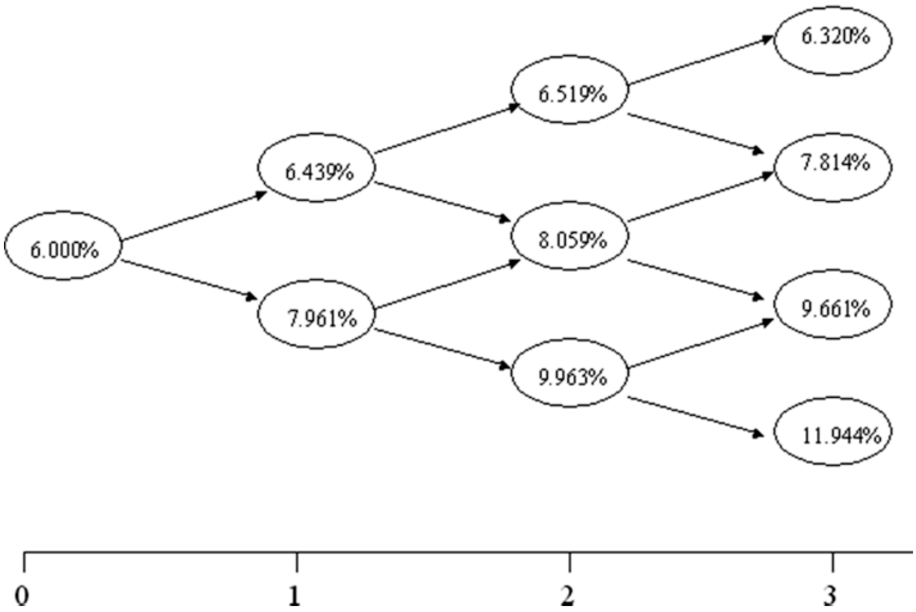

Fig. 10.2

and so on. This results in a tree with the following values
Time
Generally the rates are expressed as:

$$f_{n,1} \cdot \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot Z_n^i = 2^{n-1} \cdot f_n \quad \Rightarrow \quad f_{n,1} \quad \Rightarrow \quad f_{n,2}, \ldots, f_{n,n}$$

In this example the volatility is constant for simplicity. Generally, the volatility will change by time.

10.1.3 Calibrate the Binomial Tree

Before the tree is used it will be calibrated with the market data. This calibration process involves raising (or lowering) the estimates of the rates in the tree by an amount just sufficient so that the value for the cash flows given by the tree exactly equals the values given by the discount function. As this is done, the relationship (equation 10.2) between the different nodes must be simultaneously preserved. First, the nodes are calibrated at time 1. Once this is finished, the nodes at time 2 are calibrated, and so on. At time 1 the following must hold

$$\left(\frac{1/2}{1+f_{2,1}\cdot(t_2-t_1)}+\frac{1/2}{1+Z_2\cdot f_{2,1}\cdot(t_2-t_1)}\right)\cdot\frac{1}{1+f_{1,1}\cdot(t_1-t_0)}=P(t_0,t_2)$$

The left side of this equation is the price of a cash flow equal 1 (with equal probabilities $1/2$ given by the tree, and the right side is the price of the same cash flow given by the discount function $P(t,T)$. The discount function discount any value from $t = t_2$ to $t = t_0$, where t_0 = valuation time. This equation is solved numerically by a Van Winjgaarden-Decker-Brent method. In the previous equation, the following relationship is used:

$$f_{2,2} = Z_2 \cdot f_{2,1}$$

Therefore $f_{2,2}$ can be calculated as soon as $f_{2,1}$ is known.

At the next level, the following equation needs to be solved (note, it is not necessary to know the size of the cash flow).

$$\begin{aligned}&\frac{1}{2}\left\{\left(\frac{1/2}{1+Z_3^2\cdot f_{3,1}\cdot(t_3-t_2)}+\frac{1/2}{1+Z_3\cdot f_{3,1}\cdot(t_3-t_2)}\right)\cdot\frac{1}{1+f_{2,2}\cdot(t_2-t_1)}\right.\\&\qquad\left.+\left(\frac{1/2}{1+Z_3\cdot f_{3,1}\cdot(t_3-t_2)}+\frac{1/2}{1+f_{3,1}\cdot(t_3-t_2)}\right)\cdot\frac{1}{1+f_{1,2}\cdot(t_2-t_1)}\right\}\\&\qquad\cdot\frac{1}{1+f_{1,1}\,(t_1-t_0)}\\&\quad=P(t_0,t_3)\end{aligned}$$

Solving this equation for $f_{3,1}$ also gives $f_{3,2}$ and $f_{3,3}$ from the relations $f_{3,2} = Z_3 \cdot f_{3,1}$ and $f_{3,3} = Z_3 \cdot f_{3,2}$. Using the same method for cash flows, at all times in the tree, the tree will be fully calibrated to produce the same value as the forward rates. The new calibrated tree is now given in Fig. 10.3.

The rates in the calibrated tree are compared with the rates from the un-calibrated. The reason for the previous calibration is shown in Fig. 10.4, where the error is caused by the bond's convexity.

Notice that the present value curve is not linear. The curvature represents *convexity*. The value of the cash flow, labelled the "calculated value" as mentioned earlier, is an average of the two values V_1 and V_2. Note that this average is higher than the actual value. After the calibration, the situation is described in Fig. 10.5.

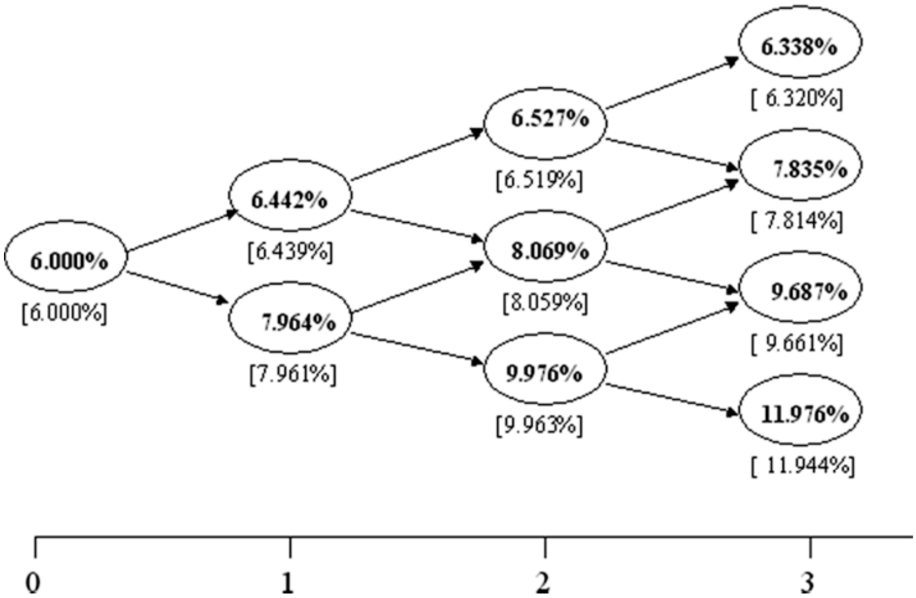

Fig. 10.3 The calibrated tree in the OAS model

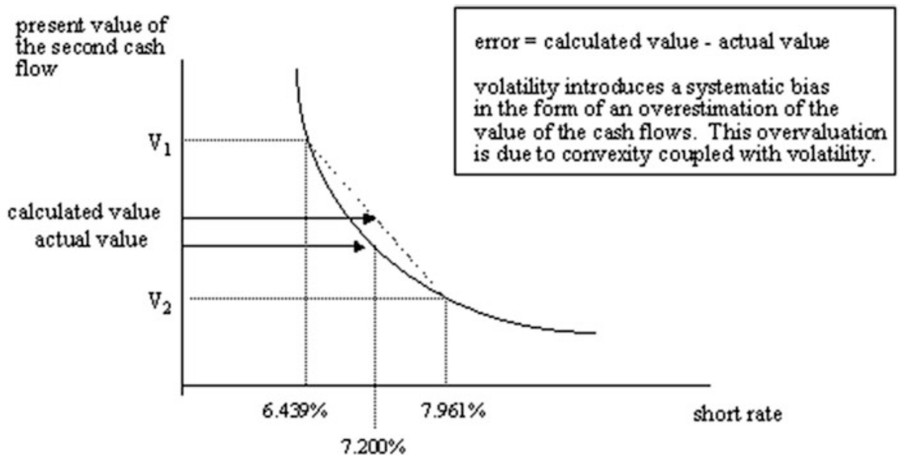

Fig. 10.4

10.1.4 Calibrate the Tree With a Spread

The calibrated binomial tree just derived is applicable to valuing a benchmark bullet instrument. Now, consider how the same, calibrated tree could be adapted to value a non-benchmark (corporate) callable bond. To simplify the analysis, it is assumed that a corporation incurs

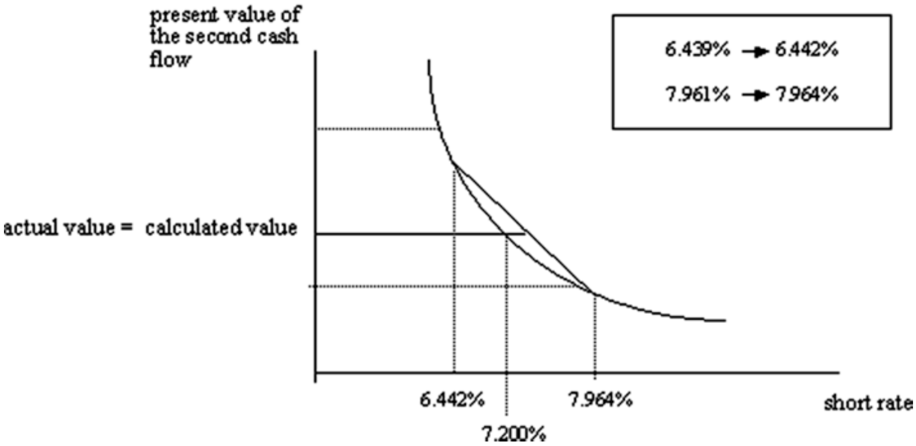

Fig. 10.5

no transaction costs either when it calls a bond or when it issues a new bond, and that it will always call a bond if it is rational to do so.

A general pricing formula at zero spread paying cash flows C_1, $C_2, \ldots C_n$ at time $T_1, T_2, \ldots T_n$ is given by

$$\Pi(0,0) = \sum_{i=1}^{n} C_i \left\{ \prod_{j=1}^{i} \frac{1}{\left(1 + f_j\right)^{T_j - T_{j-1}}} \right\}$$

With a shift $s(s \neq 0)$ in the rate f_j, the price is given as:

$$\Pi(s,0) = \sum_{i=1}^{n} C_i \left\{ \prod_{j=1}^{i} \frac{1}{\left(1 + f_j + s\right)^{T_j - T_{j-1}}} \right\}$$

If the market price Π is given, the aforementioned formula can be applied with different spreads s, until the spread that equals the market price is found. This spread is called the implied spread. When using tree models, the same spread is applied at all nodes.

Consider a 24-month corporate bond paying an annual coupon of 10.50% in two semi-annual instalments (each coupon is therefore $5.25). The bond is callable in 18 months (period 3) at $101.00. Suppose that the bond's offer price is $103.75 –this is the price at which you could buy this bond. The goal is to derive this same value with the model. To get this value, a constant spread is added to all of the rates in the tree until the value of the bond cash flows equal the price of the callable corporate bond. In the calibration procedure we replace the values of the bond with the call value if the bond can be called back

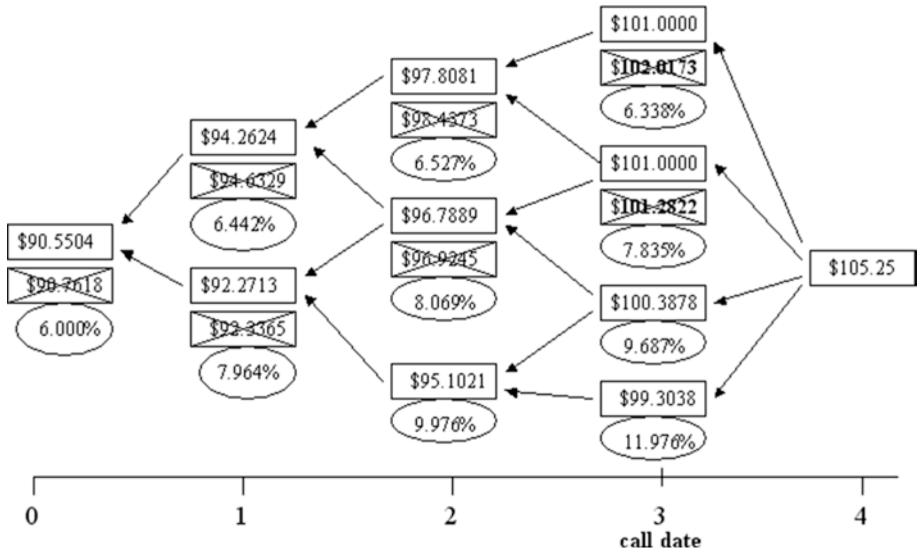

Fig. 10.6

at this time, and the value at this point exceeds the call value – this is shown for the final cash flow in Fig. 10.6.

The same is done for all cash flows, and the sum of these is taken. Then the tree is adjusted to find a new shifted tree. The correct value for the callable corporate bond gives a spread of 90.465 basis points. This spread is called the bond's *option-adjusted spread* (OAS). Essentially, interpret the OAS is interpreted as the number of basis points that must be added to each and every rate in the calibrated binomial tree of risk-free short rates to obtain a model predicted price that precisely equals the observed market value of the bond. These basis points represent the risk premium for bearing the credit risk associated with the bond. The same sort of analysis could have been performed if the bond had contained an embedded put option.

If the market price is unknown, but the size of the spread is known, this spread can be used to find a reasonable price of the callable bond. It is also possible to simulate a price to find the corresponding OAS.

10.1.5 Using the OAS Model to Value the Embedded Option

Now, the OAS can be used to determine the value of the option that is embedded in a callable bond. To accomplish this task we ask, "what

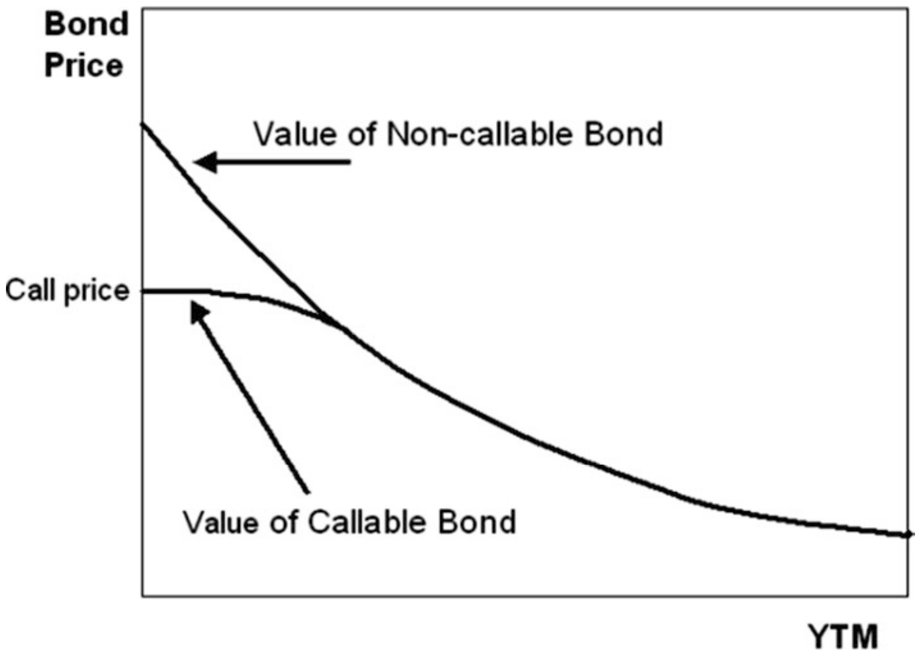

Fig. 10.7

would the value of the bond be *at the same OAS* if the bond had *not* been callable". In this case, the answer is $103.8143.

A callable bond may be viewed as a portfolio consisting of a long position in a bullet bond and a short position in a call option on a bullet bond that begins on the option's call date. Therefore,

$$B_{callable} = B_{bullet} - C_{bullet}$$
$$103.7500 = 103.8143 - C_{bullet}$$

This implies that $C_{bullet} = 0.0643$

Therefore, the option is worth $0.0643 for every $100 of par. Because of the embedded option in a callable bond, the curve, bond price as function of YTM, will differ from the curve for a non-callable (bullet) bond. This is shown in Fig. 10.7.

10.1.6 Effective Duration and Convexity

Modified duration measures the percentage bond price change for an absolute yield change. It can also be interpreted as the negative slope of the price-yield relationship. The convexity can similarly be interpreted as the curvature of the price-yield relationship. Since duration and convexity do not consider that cash flows of an interest rate security with embedded option may change due to the exercise events, they do not provide satisfactory results for instruments with embedded options. Since a callable (or putable) instrument has cash flows that differ under different interest rate scenarios, it follows that the duration is a poor measure for these instruments. The OAS approach makes it possible to get a better measure of interest rate risk. These measures are called the bond's *effective duration* or *option-adjusted duration* and the bond's *effective convexity.*

The most intuitive way to calculate an effective duration is to first calculate the callable bond's fair value using the OAS approach (as done previously). Next, it is assumed that the benchmark yield curve shifts upward by exactly one basis point. The benchmark forward rates are then re-derived, as is the calibrated binomial tree of interest rates. With the new binomial tree the upward shifted value of the callable bond is calculated. Similarly, it is then assumed that the benchmark yield curve shifts downward by exactly one basis point, and the same values are recalculated as shown before. With this tree we calculate the downward shifted value of the bond.

The effective Macaulay duration and convexity is then given by

$$\textit{Effective Duration} = \frac{P_{-} - P_{+}}{2P_0\,(\Delta y)}$$

and

$$\textit{Effective Convexity} = \frac{P_{+} + P_{-} - 2P_0}{P_0\,(\Delta y)^2}$$

where

P_{-} is the down shifted price
P_{+} the up shifted price
P_0 the un-shifted price and
Δy the shift in the yield curve

If this technique is used for the corporate bond for which we calculated an OAS of 90.465 basis points, the effective duration will be 1.745 and the effective convexity 4.045. Without the embedded option the values are 1.782 and 4.166 respectively. In this case the differences are small, but for bonds with long maturity the difference between Modified and Effective Duration can be significant. (Fig. 10.2)