Answers to Interest Rate Theory

1 (a) Consider the Vasiček model

$$dr = (b - ar)dt + \sigma dV.$$

i. The explicit solution to the stochastic differential equation above is given by

$$r(t) = e^{-at}r_0 + \int_0^t e^{-a(t-s)}bds + \int_0^t e^{-a(t-s)}\sigma dV(s).$$

We see that $r \in \mathrm{N}(\frac{b}{a} + e^{-at}(r_0 - \frac{b}{a}), \frac{\sigma^2}{2a}(1 - e^{-2at}))$.

ii. The limiting distribution is $\mathrm{N}(\frac{b}{a}, \frac{\sigma^2}{2a})$.

iii. The explicit solution found in (i) gives that

$$r_t = e^{-at}r_0 + Z,$$

where $r_0 \in \mathrm{N}(\frac{b}{a}, \frac{\sigma^2}{2a})$ has been assumed and it can be seen that $Z \in \mathrm{N}(\frac{b}{a}(1 - e^{-at}), \frac{\sigma^2}{2a}(1 - e^{-2at}))$. Since r_0 is independent of the Wiener process by assumption, it follows that r_0 and Z are independent. We thus have that

$$r(t) \in \mathrm{N}\left(e^{-at}\frac{b}{a} + \frac{b}{a}(1 - e^{-at}), e^{-2at}\frac{\sigma^2}{2a} + \frac{\sigma^2}{2a}(1 - e^{-2at})\right),$$

i.e.

$$r(t) \in \mathrm{N}\left(\frac{b}{a}, \frac{\sigma^2}{2a}\right).$$

iv. It is easily checked that

$$f(x) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2a}}}e^{-(x-\frac{b}{a})^2/2\frac{\sigma^2}{2a}},$$

satisfies

$$-\frac{\partial}{\partial x}[(b - ax)f(x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2 f(x)] = 0.$$

(b) Itô's formula applied to $Z(t) = \sqrt{Y(t)}$ gives

$$\begin{aligned} dZ &= \frac{1}{\sqrt{Y}}dY + \frac{1}{2}\left(-\frac{1}{2Y^{3/2}}(dY)^2\right) \\ &= 2aZdt + 2\sigma dW. \end{aligned}$$

The solution of this SDE is

$$Z_t = e^{2at}z_0 + \int_0^t e^{2a(t-s)}2\sigma dW_s.$$

We see that $Z \in \mathrm{N}(e^{2at}z_0, \frac{\sigma^2}{a}(e^{4at} - 1))$.

3 We have $\Pi[t;X] = p(t,T)E^T\left[r(T)|\,\mathcal{F}_t\right]$. The Girsanov kernel for the transition from Q to Q^T is given by $g(t) = v(t,T)$ where $v(t,T)$ is the bond price volatility. In our case we have an affine term structure

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

where B is given by $B_t(t,T) = -1$, $B(T) = 0$. Thus $B(t,T) = T - t$. From the affine formula above we also have

$$dp(t,T) = r(t)p(t,T)dt - p(t,T)B(t,T)\sigma dV(t),$$

so $v(t,T) = -\sigma(T-t)$. The Q^T-dynamics of r are thus given by

$$dr(t) = (\alpha + \sigma^2(t-T))ds + \sigma dW(t), r(0) = r_0.$$

where W is a Q^T-Wiener process. Thus we have

$$E^T[r(T)] = r_0 + (\alpha - \sigma^2 T)T + \frac{\sigma^2}{2}T^2 = r_0 + \alpha T - \tfrac{1}{2}\sigma^2 T^2.$$

4 (a) The Ho-Lee model possesses an affine term structure, i.e. the bond prices in this model can be written on the form

$$p(t,T) = F(t,r(t),T) = e^{A(t,T)-B(t,T)r(t)}.$$

The deterministic functions A and B solve the following ordinary differential equations

$$\begin{cases} B_t(t,T) &= -1, \\ B(T,T) &= 0, \end{cases}$$

and

$$\begin{cases} A_t(t,T) &= \phi(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) &= 0. \end{cases}$$

In order for the model to fit the initial term structure ϕ should be chosen as (see the textbook for details)

$$\phi(t) = f_T^*(0,t) + \sigma^2 t.$$

The solutions for the two ordinary differential equations are thus

$$B(t,T) = T - t,$$

and

$$A(t,T) = \int_t^T \left[f_T^*(0,s) + \sigma^2 s\right](s-T)ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}.$$

From the relation

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = -\frac{p_T(t,T)}{p(t,T)},$$

we obtain that

$$f(t,T) = B_T(t,T)r(t) - A_T(t,T).$$

After inserting the expressions for A and B this becomes

$$f(t,T) = r(t) + f^*(0,T) - f^*(0,t) + \sigma^2 t(T-t).$$

(b) $\sigma^2 t(T-t)$ is obviously linear in T for every fixed t.

5 See the textbook.

6 Heuristically we have

$$\begin{aligned} dC(t) &= -p(t,t)dt + \int_t^\infty dp(t,s)ds \\ &= -1\cdot dt + \int_t^\infty [r(t)p(t,s)dt + v(t,s)p(t,s)dW(t)]\,ds \\ &= -dt + r(t)\left[\int_t^\infty p(t,s)ds\right]dt + \left[\int_t^\infty v(t,s)p(t,s)ds\right]dW(t) \\ &= [C(t)r(t) - 1]dt + \sigma_C(t)dW(t), \end{aligned}$$

where

$$\sigma_C(t) = \int_t^\infty p(t,s)v(t,s)ds.$$

7 (a) The net payments to you at time T_n are given by

$$X_n = K\left(\exp\left\{\int_{T_{n-1}}^{T_n} r(s)ds\right\} - e^{R(T_n - T_{n-1})}\right).$$

The value at time $t=0$ of X_n is

$$\begin{aligned} \Pi[X_n] &= KE^Q\left[\exp\left\{-\int_0^{T_n} r(s)ds\right\} \times \right. \\ &\quad \left.\left(\exp\left\{\int_{T_{n-1}}^{T_n} r(s)ds\right\} - e^{R(T_n-T_{n-1})}\right)\right] \\ &= KE^Q\left[\exp\left\{-\int_0^{T_{n-1}} r(s)ds\right\}\right] \\ &\quad -KE^Q\left[\exp\left\{-\int_0^{T_n} r(s)ds\right\}\right]e^{R(T_n-T_{n-1})} \\ &= K\left[p(0,T_{n-1}) - p(0,T_n)e^{R(T_n-T_{n-1})}\right]. \end{aligned}$$

The swap rate is thus given as the solution to the following equation

$$\sum_{n=1}^{M}\{p(0,T_{n-1}) - p(0,T_n)e^{R(T_n-T_{n-1})}\} = 0.$$

If in particular $T_n = n\Delta$, then

$$\sum_{n=0}^{M-1} p(0,T_n) = e^{R\Delta}\sum_{n=1}^{M} p(0,T_n),$$

i.e.

$$R = \frac{1}{\Delta} \ln \left\{ \frac{\sum_{n=0}^{M-1} p(0,T_n)}{\sum_{n=1}^{M} p(0,T_n)} \right\}.$$

(b) The net payments to you at time T_n are given by

$$X_n = K\left(e^{R_{n-1}(T_n - T_{n-1})} - e^{R(T_n - T_{n-1})}\right),$$

where

$$R_{n-1} = -\frac{1}{T_n - T_{n-1}} \ln\{p(T_{n-1}, T_n)\}.$$

Thus we have that

$$X_n = K\left(\frac{1}{p(T_{n-1}, T_n)} - e^{R(T_n - T_{n-1})}\right),$$

and the value at time $t = 0$ of X_n is

$$\begin{aligned}
\Pi[X_n] &= KE^Q\left[\exp\left\{-\int_0^{T_n} r(s)ds\right\} \times \left(\frac{1}{p(T_{n-1}, T_n)} - e^{R(T_n - T_{n-1})}\right)\right] \\
&= KE^Q\left[\exp\left\{-\int_0^{T_{n-1}} r(s)ds\right\} \frac{\exp\left\{-\int_{T_{n-1}}^{T_n} r(s)ds\right\}}{p(T_{n-1}, T_n)}\right] \\
&\quad -KE^Q\left[\exp\left\{-\int_0^{T_n} r(s)ds\right\}\right] e^{R(T_n - T_{n-1})} \\
&= KE^Q\left[E^Q\left[\exp\left\{-\int_0^{T_{n-1}} r(s)ds\right\} \frac{\exp\left\{-\int_{T_{n-1}}^{T_n} r(s)ds\right\}}{p(T_{n-1}, T_n)} \middle| \mathcal{F}_{T_{n-1}}\right]\right] \\
&\quad -KE^Q\left[\exp\left\{-\int_0^{T_n} r(s)ds\right\}\right] e^{R(T_n - T_{n-1})} \\
&= KE^Q\left[\frac{\exp\left\{-\int_0^{T_{n-1}} r(s)ds\right\}}{p(T_{n-1}, T_n)} E^Q\left[\exp\left\{-\int_{T_{n-1}}^{T_n} r(s)ds\right\} \middle| \mathcal{F}_{T_{n-1}}\right]\right] \\
&\quad -KE^Q\left[\exp\left\{-\int_0^{T_n} r(s)ds\right\}\right] e^{R(T_n - T_{n-1})} \\
&= KE^Q\left[\frac{\exp\left\{-\int_0^{T_{n-1}} r(s)ds\right\}}{p(T_{n-1}, T_n)} p(T_{n-1}, T_n)\right] \\
&\quad -KE^Q\left[\exp\left\{-\int_0^{T_n} r(s)ds\right\}\right] e^{R(T_n - T_{n-1})} \\
&= K\left[p(0, T_{n-1}) - p(0, T_n) e^{R(T_n - T_{n-1})}\right].
\end{aligned}$$

This coincides with the value found in (a) and we obtain the same swap rate as in (a).

(c) In a continuous model the net payments to you in the interval $[t, t+dt]$ are given by

$$Kr(t)dt - KRdt.$$

The value of the total payment stream is

$$\begin{aligned}
\Pi &= KE^Q\left[\int_0^T \exp\left\{-\int_0^t r(s)ds\right\}[r(t)-R]dt\right] \\
&= KE^Q\left[\int_0^T \exp\left\{-\int_0^t r(s)ds\right\}r(t)dt\right] \\
&\quad -KRE^Q\left[\int_0^T \exp\left\{-\int_0^t r(s)ds\right\}dt\right] \\
&= K\int_0^T -\frac{d}{dt}E^Q\left[\exp\left\{-\int_0^t r(s)ds\right\}\right]dt \\
&\quad -KR\textstyle\int_0^T E^Q\left[\exp\left\{-\int_0^t r(s)ds\right\}\right]dt \\
&= K\left\{1-E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\right]\right\} \\
&\quad -KR\int_0^T E^Q\left[\exp\left\{-\int_0^t r(s)ds\right\}\right]dt \\
&= K[1-p(0,T)] - KR\int_0^T p(0,t)dt.
\end{aligned}$$

We see that

$$R = \frac{[1-p(0,T)]}{\int_0^T p(0,t)dt}.$$

Answers to Change of Numeraire

1 (a) $L^T(t) = \dfrac{p(t,T)}{p(0,T)B(t)}$ (see the textbook for details).

(b) Itô's formula applied to the expression for $L^T(t)$ derived in (a) gives

$$dL^T(t) = v(t,T)L^T(t)dV(t).$$

(c) The model possesses an affine term structure, i.e. the bond prices in this model can be written on the form

$$p(t,T) = F(t,r(t),T) = e^{A(t,T)-B(t,T)r(t)}.$$

The deterministic functions A and B solve the following ordinary differential equations

$$\begin{cases} B_t(t,T) = -1, \\ B(T,T) = 0, \end{cases}$$

and

$$\begin{cases} A_t(t,T) = \alpha B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) = 0. \end{cases}$$

The solution of the ODE for B is $B(t,T) = T-t$. Itô's formula applied to $p(t,T) = \exp\{A(t,T) - B(t,T)r(t)\}$ gives us the diffusion term we need in order to write down the Q-dynamics of $p(t,T)$ (we already know that the drift should be equal to the short rate)

$$dp(t,T) = r(t)p(t,T)dt - \sigma(T-t)p(t,T)dV.$$

This means that

$$v(t,T) = -\sigma(T-t).$$

Via Girsanov's Theorem we now have that the dynamics of r under Q^T are given by

$$dr(t) = [\alpha - \sigma(T-t)]dt + \sigma dV^T(t),$$

where V^T is a Q^T-Wiener process. Given that $r(t) = r$, it follows that (under Q^T)

$$r(T) = r + \int_t^T [\alpha - \sigma(T-s)]ds + \int_t^T \sigma dV^T(s).$$

We see that $r(T) \in N[r + \alpha(T-t) + \frac{\sigma}{2}(T-t)^2, \sigma\sqrt{T-t}]$. The price of the contract $X = r^2(T)$ is given by

$$\Pi(t) = p(t,T)E^T\left[r^2(T)\middle| \mathcal{F}_t\right].$$

Since $E[X^2] = Var[X] + E^2[X]$, the expectation is easily found to be

$$E^T\left[r^2(T)\middle| \mathcal{F}_t\right] = \sigma^2(T-t) + m^2,$$

where $m = r + \alpha(T-t) + \frac{\sigma}{2}(T-t)^2$.

2 (a) Since $\Pi(t)/Y(t)$ is a Q^*-martingale we have

$$\frac{\Pi[t;X]}{Y(t)} = E^*\left[\frac{\Pi[T;X]}{Y(T)}\middle| \mathcal{F}_t\right] = E^*\left[\frac{X}{Y(T)}\middle| \mathcal{F}_t\right],$$

i.e

$$\Pi[t;X] = Y(t)E^*\left[\frac{X}{Y(T)}\middle| \mathcal{F}_t\right].$$

(b) With $X = ZY(T)$ and $T = 0$ we get, using (a) and known results, that

$$e^{-rT}E^Q[ZY(T)] = \Pi[0;X] = Y(0)E^*\left[\frac{ZY(T)}{Y(T)}\right]$$
$$= Y(0)E^*[Z],$$

that is

$$E^*[Z] = E^Q\left[Z\frac{Y(T)}{B(T)Y(0)}\right].$$

Our guess would then be that the Radon-Nikodym derivative is given by

$$L(T) = \frac{Y(T)}{B(T)Y(0)}.$$

The likelihood process is then

$$L(t) = E^Q[L(T)|\mathcal{F}_t] = E^Q\left[\frac{Y(T)}{B(T)Y(0)}\middle| \mathcal{F}_t\right] = \frac{Y(t)}{B(t)Y(0)},$$

since, by definition, $Y(T)/B(T)$ is a Q-martingale.

It remains to show that $S(t)/Y(t)$ is a Q^*-martingale. We have that

$$E^*\left[\left.\frac{S(T)}{Y(T)}\right|\mathcal{F}_t\right] = \frac{E^Q\left[L(T)\frac{S(T)}{Y(T)}\Big|\mathcal{F}_t\right]}{E^Q\left[L(T)|\mathcal{F}_t\right]}$$

$$= \frac{E^Q\left[\frac{Y(T)}{B(T)Y(0)}\frac{S(T)}{Y(T)}\Big|\mathcal{F}_t\right]}{L(t)} = \frac{E^Q\left[\frac{S(T)}{B(T)Y(0)}\Big|\mathcal{F}_t\right]}{L(t)}$$

$$= \frac{\frac{S(t)}{B(t)Y(0)}}{\frac{Y(t)}{B(t)Y(0)}} = \frac{S(t)}{Y(t)}.$$

(c) The Q-dynamics of Y are

$$dY = rYdt + \sigma YdV.$$

From (b) and Itô's formula it follows that

$$dL(t) = d\left(\frac{Y(t)}{B(t)Y(0)}\right) = \frac{1}{Y(0)}d\left(\frac{Y(t)}{B(t)}\right)$$

$$= \frac{1}{Y(0)}\sigma\frac{Y(t)}{B(t)}dV(t) = \sigma L(t)dV(t).$$

(d) Under Q we have the following dynamics, using Girsanov's Theorem

$$\begin{aligned} dY &= rYdt + \sigma YdV, \\ dS &= rSdt + \delta SdV^*, \end{aligned}$$

where V and V^* are independent Q-Wiener processes. Girsanov's Theorem gives that the Q^*-dynamics are

$$\begin{aligned} dY &= (r+\sigma)Ydt + \sigma YdV^{**}, \\ dS &= rSdt + \delta SdV^*, \end{aligned}$$

where V^* (the same process as above) and V^{**} are independent Q^*-Wiener processes. From this and (a) we obtain

$$\Pi[t;X] = Y(t)E^*\left[\left.\frac{X}{Y(T)}\right|\mathcal{F}_t\right] = Y(t)E^*\left[\left.\frac{Y(T)S(T)}{Y(T)}\right|\mathcal{F}_t\right]$$

$$= Y(t)E^*\left[S(T)|\mathcal{F}_t\right] = Y(t)E^Q\left[S(T)|\mathcal{F}_t\right] = Y(t)S(t)e^{r(T-t)}.$$

3 (a) We start by going through the steps in the hint.

i. Itôs formula applied to $F^T(t,r_t)/S_0(t)$ gives

$$\begin{aligned} d\left(\frac{F^T}{S_0}\right) &= -\frac{F^T}{S_0^2}dS^0 + \frac{1}{S^0}dF^T = \\ &= -r\frac{F^T}{S_0}dt + \frac{1}{S_0}\left(F_t^T + aF_r^T + \frac{1}{2}b_1^2F_{rr}^T + \frac{1}{2}b_2^2F_{rr}^T\right)dt \\ &\quad + \frac{1}{S_0}b_1F_r^TdW^1 + \frac{1}{S_0}b_2F_r^TdW^2 \end{aligned}$$

Since $F^T(t,r_t)/S_0(t)$ is a martingale, the drift term has to be zero, which gives

$$F_t^T + aF_r^T + \frac{1}{2}b_1^2F_{rr}^T + \frac{1}{2}b_2^2F_{rr}^T - rF^T = 0.$$

The boundary value is of course $F(T,r,T) = 1$.

ii. Just insert the proposed term structure into the equation and check that things equates.

Now that we know that $p(t,T) = \exp\{A(t,T) - B(t,T)r(t)\}$, Itô's formula gives

$$\begin{aligned} dp(t,T) &= r_t p(t,T)dt - b_1(t)B(t,T)p(t,T)dW_t^1 \\ &\quad -b_2(t)B(t,T)p(t,T)dW_t^2. \end{aligned}$$

(b) Note that the bond issued by the firm can be seen as a contingent T-claim, which, at time T, pays

$$\min\left\{1, \frac{V_T}{K}\right\}.$$

By using the pricing formula in Proposition XXII.2 we obtain

$$\begin{aligned} p(0,T) - u(0,T) &= p(0,T) - p(0,T)E^T\left[\min\left\{1, \frac{V_T}{K}\right\}\right] \\ &= p(0,T)E^T\left[1 - \min\left\{1, \frac{V_T}{K}\right\}\right] \\ &= \frac{p(0,T)}{K}E^T\left[\max\{K - V_T, 0\}\right] \\ &= \frac{p(0,T)}{K}E^T\left[\max\left\{K - \frac{V_T}{p(T,T)}, 0\right\}\right]. \end{aligned}$$

Since $V(t)/p(t,T)$ is a Q^T-martingale the drift and thereby the interest rate must equal zero. Since the change of measure is a Girsanov transformation it does not affect the volatility, and Itô's formula applied to $V(t)/p(t,T)$ under Q gives

$$d\left(\frac{V}{p^T}\right) = \ldots dt + (b_1 B^T + \sigma_V^1)\frac{V}{p^T}dW_t^1 + (b_2 B^T + \sigma_V^2)\frac{V}{p^T}dW_t^2.$$

The price of the bond issued by the firm can thus be computed as the price of a European put option on $V(t)/p(t,T)$, with strike price K, interest rate zero and volatilities according to the above.

Remark: If one prefers, one can use the following equality in distribution, when computing the expectation

$$\begin{aligned} d\left(\frac{V}{p^T}\right) &= \ldots dt + (b_1 B^T + \sigma_V^1)\frac{V}{p^T}dW_t^1 + (b_2 B^T + \sigma_V^2)\frac{V}{p^T}dW_t^2 \\ &= \ldots dt + \sqrt{(b_1 B^T + \sigma_V^1)^2 + (b_2 B^T + \sigma_V^2)^2}\frac{V}{p^T}dW^3, \end{aligned}$$

where W^3 is a Wiener process.

4 From the lecture notes we have that

$$c(t,T,K,S) = p(t,T)\int_{-\infty}^{\infty} \max\left\{e^{A(T,S)-B(T,S)z} - K, 0\right\}\varphi(z)dz.$$

Here φ denotes the density function of the $\mathrm{N}\left(f(t,T), \sigma^2(T-t)\right)$ distribution, and A and B solve the following ordinary differential equations

$$\begin{cases} B_t(t,T) &= -1, \\ B(T,T) &= 0, \end{cases}$$

and

$$\begin{cases} A_t(t,T) &= \phi(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) &= 0. \end{cases}$$

The solutions are given by

$$\begin{cases} B(t,T) &= T-t, \\ A(t,T) &= \int_t^T \phi(s)(s-T)ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}. \end{cases}$$

Take a closer look at the integral

$$p(t,T)\int_{-\infty}^{\infty} \max\left\{e^{A(T,S)-B(T,S)z} - K, 0\right\}\varphi(z)dz =$$

$$p(t,T)\left[0\cdot Q^T\left(Z > \frac{-\ln K + A(T,S)}{B(T,S)}\right) + \right.$$

$$\left. + \int_{-\infty}^{\frac{-\ln K+A(T,S)}{B(T,S)}} \left(e^{A(T,S)-B(T,S)z} - K\right)\varphi(z)dz\right] =$$

$$p(t,T)\int_{-\infty}^{\frac{-\ln K+A(T,S)}{B(T,S)}} e^{A(T,S)-B(T,S)z}\varphi(z)dz -$$

$$-p(t,T)KQ^T\left(Z \le \frac{-\ln K + A(T,S)}{B(T,S)}\right).$$

The probability in the last term can be written as

$$Q^T\left(Z \le \frac{-\ln K + A(T,S)}{B(T,S)}\right) =$$

$$Q^T\left(\frac{Z - f(t,T)}{\sigma\sqrt{(T-t)}} \le \frac{-\ln K + A(T,S) - f(t,T)B(T,S)}{\sigma\sqrt{(T-t)}B(T,S)}\right) =$$

$$N\left(\frac{-\ln K + A(T,S) - f(t,T)B(T,S)}{\sigma\sqrt{(T-t)}B(T,S)}\right).$$

The argument of the cumulative distribution function can be rewritten as

$$\frac{-\ln K + A(T,S) - f(t,T)B(T,S)}{\sigma\sqrt{(T-t)}B(T,S)} =$$

$$\frac{-\ln K + A(T,S) - [B_T(t,T)r(t) - A_T(t,T)]B(T,S)}{\sigma\sqrt{(T-t)}B(T,S)} =$$

$$\frac{-\ln K + A(T,S) + A_T(t,T)B(T,S) - B_T(t,T)B(T,S)r(t)}{\sigma\sqrt{(T-t)}B(T,S)} =$$

$$\frac{-\ln K + A(t,S) - A(t,T) - \frac{1}{2}\sigma^2(T-t)B^2(T,S) - [B(t,S) - B(t,T)]r(t)}{\sigma\sqrt{(T-t)}B(T,S)}.$$

In the first equality we have used that

$$f(t,T) = B_T(t,T)r(t) - A_T(t,T).$$

In the last equality we have used that

$$B(t,S) - B(t,T) = B_T(t,T)B(T,S), \tag{1}$$

which is easy to check, and that

$$A(t,S) - A(t,T) = A(T,S) + A_T(t,T)B(T,S) + \frac{1}{2}\sigma^2(T-t)B^2(T,S), \tag{2}$$

which we now show.

$$\int_t^S \phi(s)(s-S)ds + \int_t^S \frac{1}{2}\sigma^2(S-s)^2 ds - \int_t^T \phi(s)(s-T)ds +$$

$$+ \int_t^T \frac{1}{2}\sigma^2(T-s)^2 ds =$$

$$\int_T^S \phi(s)(s-S)ds + \int_T^S \frac{1}{2}\sigma^2(S-s)^2 ds +$$

$$+ \int_t^T \left\{\phi(s)(T-S) + \frac{\sigma^2}{2}\left[(S-s)^2 - (T-s)^2\right]\right\} ds =$$

$$A(T,S) + B(T,S)\int_t^T \{\phi(s) + \sigma^2(T-s)\}ds + \frac{\sigma^2}{2}\int_t^T (S-T)^2 ds =$$

$$A(T,S) + B(T,S)A_T(t,T) + \tfrac{1}{2}\sigma^2 B^2(T,S).$$

The argument can thus be written as

$$\frac{-\ln K + A(t,S) - A(t,T) - \frac{1}{2}\sigma^2(T-t)B^2(T,S) - [B(t,S) - B(t,T)]r(t)}{\sigma\sqrt{(T-t)}B(T,S)} =$$

$$\frac{1}{\sigma\sqrt{(T-t)}B(T,S)}\ln\left\{\frac{p(t,S)}{p(t,T)K}\right\} - \frac{1}{2}\sigma\sqrt{(T-t)}B(T,S) =$$

$$d - \sigma_p.$$

Now to the integral

$$p(t,T)\int_{-\infty}^{\frac{-\ln K+A(T,S)}{B(T,S)}} e^{A(T,S)-B(T,S)z}\varphi(z)dz.$$

By completing the square in the exponent, the integral can be written as

$$p(t,T)e^{A(T,S)-f(t,T)B(T,S)+\frac{1}{2}\sigma^2B^2(T,S)}\int_{-\infty}^{\frac{-\ln K+A(T,S)}{B(T,S)}} \psi(z)dz =$$

$$e^{A(t,T)-B(t,T)r(t)+A(T,S)-f(t,T)B(T,S)+\frac{1}{2}\sigma^2B^2(T,S)}\times$$

$$\times Q^T\left(Z\leq\frac{-\ln K+A(T,S)}{B(T,S)}\right)=$$

$$e^{A(t,T)-B(t,T)r(t)+A(T,S)-f(t,T)B(T,S)+\frac{1}{2}\sigma^2B^2(T,S)}\times$$

$$\times N\left(\frac{-\ln K+A(T,S)-f(t,T)B(T,S)+\sigma^2(T-t)B^2(T,S)}{\sigma\sqrt{(T-t)}B(T,S)}\right)$$

where ψ denotes the density function of N $(f(t,T)-\sigma^2(T-t)B(T,S),\sigma^2(T-t))$. It is easy to see that the argument of the distribution function is given by the same argument studied earlier plus $\sigma\sqrt{(T-t)}B(T,S)=\sigma_p$, i.e. the argument is given by

$$d-\sigma_p+\sigma_p=d.$$

By using the relations (1) and (2) and $f(t,T)=B_T(t,T)r(t)-A_T(t,T)$ you can also see that

$$e^{A(t,T)-B(t,T)r(t)+A(T,S)-f(t,T)B(T,S)+\frac{1}{2}\sigma^2B^2(T,S)}=$$

$$=e^{A(t,S)-B(t,S)r(t)}=p(t,S).$$

We are done!

5 We obtain

$$\begin{aligned}\Pi[0;X] &= E^Q[B^{-1}[S_T-K]I\{S_T\geq K\}]\\ &= E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}S_TI\{S_T\geq K\}\right]\\ &\quad -KE^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}I\{S_T\geq K\}\right].\end{aligned}$$

Now change to Q^S in the first term and Q^T in the second and you will obtain the formula from the exercise.