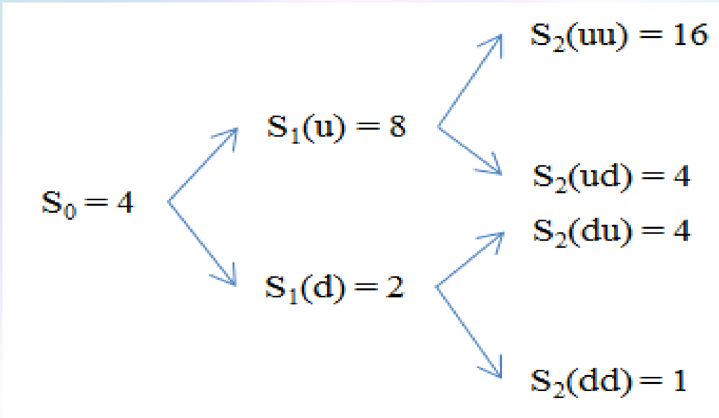
Stopping times and American options

For an American contract with a value process $V_n = g(S_n)$ we define the backward recursion as

$$\begin{cases} V_n(x) = g(x) \\ V_k(x) = \max\left\{ \dfrac{1}{1+r}\left(pV_{k+1}(ux) + qV_{k+1}(dx) \right), g(x) \right\} \end{cases}$$

Let us study a two-period binomial tree for an American put option med with $S_0 = 4$, $u = 2$, $d = ½$, $p = q = ½$ and $r = ¼$ with a strike price $X = 5$.



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At maturity we have the value: $V_2 = \left(5 - S_k\right)^+$

We have: $V_{uu} = 0$, $V_{ud} = V_{du} = 1$ and $V_{dd} = 4$. The tree gives us:

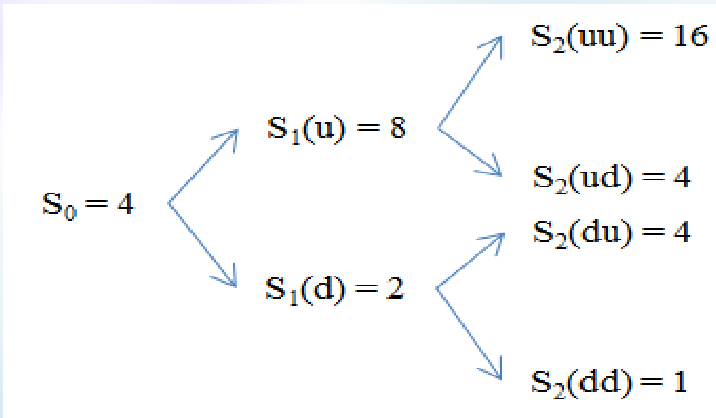
$$V_u = \max\left\{\frac{1}{1+r}\left[pV_{uu} + qV_{ud}\right], (5-8)^+\right\} = \max\left\{\frac{4}{5}\left[\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1\right], 0\right\} = 0.40$$

$$V_d = \max\left\{\frac{1}{1+r}\left[pV_{ud} + qV_{dd}\right], (5-2)^+\right\} = \max\left\{\frac{4}{5}\left[\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4\right], 3\right\} = \max\left\{2, 3\right\} = 3$$

$$V = \frac{1}{1+r}\left[pV_u + qV_d\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 0,4 + \frac{1}{2}\cdot 3\right] = 1.36$$

$$\Delta_{k-1} = \frac{V_k(u) - V_k(d)}{S_k(u) - S_k(d)}$$

$$\Delta_0 = \frac{0.4 - 3}{8 - 2} = -0.43$$



Stopping times and American options

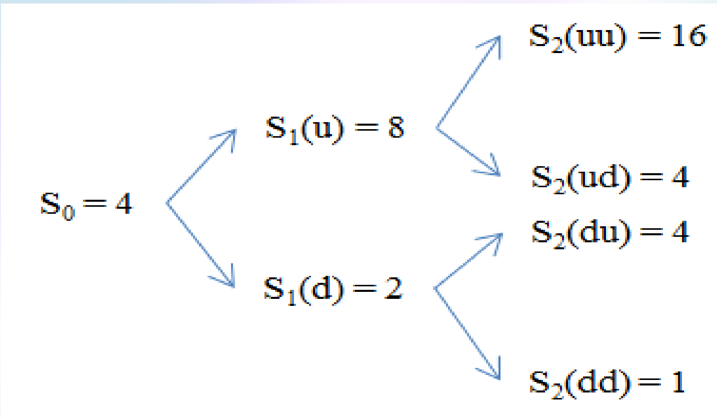
$$1 = V_{du} = S_2(du)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d))$$

$$= 4\Delta_1 + \frac{5}{4}\left(3 - \Delta_1 \cdot 2\right) \quad \Rightarrow \quad \Delta_1(d) = \frac{1 - 15/4}{4 - 5/2} = \frac{-11/4}{3/2} = -\frac{11}{6} = -1.83$$

$X_1(d) = V_d = 3$ (= 5 – 2) due to early exercise, $V_{du} = 5 - 4 = 1$, $V_{dd} = 5 - 1 = 4$

$$4 = V_{dd} = S_2(dd)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d))$$

$$= \Delta_1 + \frac{5}{4}\left(3 - 2\Delta_1\right) \quad \Rightarrow \quad \Delta_1(d) = \frac{4 - 15/4}{1 - 5/2} = \frac{1/4}{-3/2} = -\frac{1}{6} = -0.16$$



Stopping times and American options

If this was a European option $X_1(d) = S_1(d) = \frac{4}{5}\left[\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 4\right] = 2$ and

$$V_u = \frac{1}{1+r}\left[pV_{uu} + qV_{ud}\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1\right] = 0.40$$

$$V_d = \frac{1}{1+r}\left[pV_{ud} + qV_{dd}\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4\right] = 2$$

The American was 3

$$V = \frac{1}{1+r}\left[pV_u + qV_d\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 0,4 + \frac{1}{2}\cdot 2\right] = 0.96$$

The American was 1.36

SO

$$1 = V_{du} = S_2(du)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d)) = 4\cdot\Delta_1 + \frac{5}{4}\left(2 - 2\cdot\Delta_1\right)$$

$$=> \Delta_1\left(4 - 2.5\right) = 1 - 2.5 = -1.5$$

$$=> \Delta_1 = -1.0$$

$$4 = V_{dd} = S_2(dd)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d)) = 1\cdot\Delta_1 + \frac{5}{4}\left(2 - 2\cdot\Delta_1\right)$$

$$=> \Delta_1\left(1 - 2.5\right) = 4 - 2.5 = -1.5$$

$$=> \Delta_1 = -1.0$$

Stopping times and American options

The value of a hedged portfolio with an American option is given by:

$$\begin{aligned} X_{k+1} &= S_{k+1}\Delta_k + (1+r)(X_k - \Delta_k S_k - C_k) \\ &= (1+r)X_k + \Delta_k\left(S_{k+1} - (1+r)S_k - (1+r)C_k\right) \end{aligned}$$

where C_k is the part that can be consumed at time $t = k$.

<u>Properties</u>:

- The discounted portfolio value is a super martingale.
- The value satisfy $X_k \geq g(S_k),\ k = 0, 1, ..., n.$
- The value process is the process with the lowest value with these properties.

Stopping times and American options

<u>Question</u>: When do we consume?
<u>Answer</u>: If (in the binomial model)

$$E\left[(1+r)^{-(k+1)}V_{k+1}(S_{k+1})\,|\,\mathcal{F}_k\right] < (1+r)^{-k}V_k(S_k) \quad \Rightarrow$$

$$\frac{1}{1+r}E\left[V_{k+1}(S_{k+1})\,|\,\mathcal{F}_k\right] < V_k(S_k)$$

If the holder of the option doesn't exercise, then we can consume and close the gap! In that case, when $X_k = V_k(S_k)$ for all values of k and where

$$\begin{cases} V_n(x) = g(x) \\ V_k(x) = \max\left\{\dfrac{1}{1+r}\left(pV_{k+1}(ux) + qV_{k+1}(dx)\right), g(x)\right\} \end{cases}$$

Stopping times and American options

In the previous example e.g., $V_1(S_1(u)) = 3$, $V_2(S_2(ud)) = 1$, $V_2(S_2(uu)) = 4$,we get

$$\frac{1}{1+r} E\left[V_2(S_2) \mid \mathcal{F}_1\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4\right] = 2$$

If the holder don't exercise at $t = 1$ we can consume one cash unit (the difference of an American and a European) and hedge as

$$\Delta_k = \frac{V_{k+1}(uS_k) - V_{k+1}(dS_k)}{(u-d)S_k}$$

As we can see, from the holder's point of view, it is optimal to exercise when $V_k(S_k) = g(S_k)$.

Stopping times

Definition: Given the probability space $(\Omega, \mathcal{F}, P)$ and the filtration $\{\mathcal{F}_k\}_{k=0}^{n}$ of $\mathcal{F}$ we define the **stopping time** as a stochastic variable $\tau : \Omega \to \{0, 1, ..., n\} \cup \{\infty\}$ such as

$$\{\omega \in \Omega;\ \tau(\omega) = k\} \in \mathcal{F}_k \ \forall\ k = 0, 1, ..., n, \infty$$

Example – Stopping time

We define (from the tree above)

$$\tau(\omega) = \min\{k \mid V_k(S_k) = (5 - S_k)^+\}$$

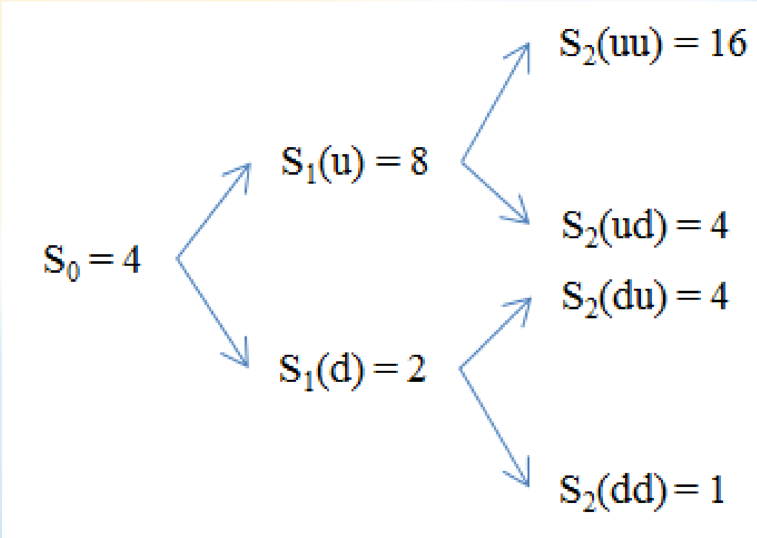
This stopping time is the time when the option value for the first time is equal to the instantaneous value. This time is the optimal time to exercise the option. A stopping time is characterized by the fact that we at every time $t < \tau$ can decide if τ has occurred or not, based on the information we really have at time t. Remark:

$$\tau(\omega) = \begin{cases} 1 & \text{if } \omega = A_d \\ 2 & \text{if } \omega = A_u \end{cases}$$

$$\{\omega : \tau(\omega) = 0\} = \emptyset \in \mathcal{F}_0$$
$$\{\omega : \tau(\omega) = 1\} = A_d \in \mathcal{F}_1$$
$$\{\omega : \tau(\omega) = 2\} = A_u \in \mathcal{F}_2$$

Markov processes

We start by studying a European lookback option with values S_0 = *4*, *u* = *2*, *d* = *½*, *p* = *q* = *½* and *r* = *¼* with a strike price *K* = *5* with a two-period binomial model:



The value of the lookback option is given by:

$$V_2 = \max_{0 \le t \le 2}\left(S_k - 5, 0\right)$$ (i.e., 11, 3 and 0)

Markov processes

We study the evolution backwards to calculate the value, thereby the name lookback. We have: $V_{uu} = 11$, $V_{ud} = 3$, $V_{du} = 0$ and $V_{dd} = 0$. (Remark $V_{ud} \neq V_{du}$). By travelling backwards in the tree we get:

$$V_u = \frac{1}{1+r}\left[pV_{uu} + qV_{ud}\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 11 + \frac{1}{2}\cdot 3\right] = 5.60$$

$$V_d = 0$$

$$V = \frac{4}{5}\cdot\frac{1}{2}\cdot 5.60 = 2.24$$

Remark American 1.36 and European 0.96

with

$$\Delta_{t-1} = \frac{V_t(u) - V_t(d)}{S_t(u) - S_t(d)}$$

we get $\Delta_0 = (5.6 - 0.0)/(8 - 2) = 0.93$, $\Delta_1(u) = (11.0 - 3.0)/(16 - 4) = 0.67$ and $\Delta_1(d) = 0$. If we now sell one option at $X_0 = 2.24$ and hedge us with Δ_0 shares we get

Markov processes

$$X_1(u) = \Delta_0 S_1(u) + (1+r)(X_0 - \Delta_0 S_0)$$
$$= 0.93 \cdot 8 + (1+0.25)(2.24 - 0.93 \cdot 4)$$
$$= 5.60$$

$$X_1(d) = \Delta_0 S_1(d) + (1+r)(X_0 - \Delta_0 S_0)$$
$$= 0.93 \cdot 2 + (1+0.25)(2.24 - 0.93 \cdot 4)$$
$$= 0$$

$$X_2(uu) = \Delta_1(u) S_2(uu) + (1+r)(X_1(u) - \Delta_1(u) S_1(u))$$
$$= 0.67 \cdot 16 + (1+0.25)(5.60 - 0.67 \cdot 8)$$
$$= 11.0$$

$$X_2(ud) = \Delta_1(u) S_2(ud) + (1+r)(X_1(u) - \Delta_1(u) S_1(u))$$
$$= 0.67 \cdot 4 + (1+0.25)(5.60 - 0.67 \cdot 8)$$
$$= 3.0$$

Markov processes

A general problem:

For a model with n periods, we have in Ω 2^n elements giving 2^n equations. For a three months option we have 66 trading days and with a period length of one day we get $2^{66} \approx 7*10^{19}$ equations.

Markov processes

Solution:

We can solve this in three ways:

1. By simulations and averaging.
2. Approximate in continuous time. This gives a PDE-theory.
3. Using a Markov structure.

What we are doing in the binomial model is exactly 3.) above. Instead or four values at $n = 2$ (V_{uu}, V_{ud}, V_{du} and V_{dd}) we have three, because of $V_{ud} = V_{du}$. This gives us $n + 1$ equations instead of 2^n.