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Examination in MMA707 & MT1410 Analytical Finance I
Wednesday 24 of October 2007, 14:30 – 18:30

Examiner: Jan Röman, phone 0708-606 306
You may use: Pencil, ruler, rubber gum and calculator.
General direction: The solution should be well motivated and readable. All notations must be explained.
Remark: Write your national registration number (personnummer) and the number of pages on the first page. Write only one solution on per sheet. Use page numbers and write your name on all pages.

Remark2: There are one extra problem (the last one). You don't have to make it. But if you are close to a limit I will consider it.

Good Luck!!

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1. Prove the put-call relationship for American options (current time is t = 0 maturity at T):

$$S - K \le C_A - P_A \le S - Ke^{-rT}$$

…………………………………………………………………..…………………. (3p)

2. Assume a one-period financial market model with three securities on the probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = P(\Omega)$ and $P(\omega_i) > 0$ $i = 1, 2, 3$. The current prices of the securities are $S(0) = (S_0(0), S_1(0), S_2(0)) = (100, 150, \alpha)$. At time $t = 1$ the prices are given by the following matrix:

$$S(1) = \begin{pmatrix} S_1(1,\omega_1) & S_1(1,\omega_2) & S_1(1,\omega_3) \\ S_2(1,\omega_1) & S_2(1,\omega_2) & S_4(1,\omega_3) \\ S_3(1,\omega_1) & S_3(1,\omega_2) & S_3(1,\omega_3) \end{pmatrix} = \begin{pmatrix} 110 & 110 & 110 \\ 154 & 198 & 143 \\ 176 & 220 & 143 \end{pmatrix}$$

(a) Name an equivalent characterization to freedom of arbitrage in single period market models.
(b) What are the possible values for α, so that the market remains arbitrage-free?
(c) Assume that $\alpha = 160$. Calculate an equivalent martingale measure EMM with the bond as numéraire.
(d) Calculate the price of the asset with payoff-vector $C(1) = (22, 66, 0)$ …………………………………………………………………………..………….(4p)

3. Assume a standard 3-period CRR binomial model. The price of the stock is currently \$100. The risk-free interest rate with continuous compounding is 6% per annum. Over the next three 4 month periods, the stock is expected to go up by 8% or go down by 7% in each period.

   (a) What is the value of a one-year European call with strike price \$103?
   (b) What is the value of a one-year European put with strike price \$103?
   (c) Verify the Put-Call parity for the European call and the European put.

…………………………………………………………………………………..(3p)

4. Consider a financial market in which the Black-Scholes formula for a European call option holds. The risk-free interest rate (cont. compounding) is *r*. The underlying stock has value *S* with volatility σ. For a European call with strike *K* and maturity T, show that the following relations hold:

$$
\begin{aligned}
\Delta &= \frac{\partial C}{\partial S} = N(d_1)\\
\Gamma &= \frac{\partial C}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}\\
\Theta &= \frac{\partial C}{\partial t} = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)\\
\rho &= \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2)\\
\nu &= \frac{\partial C}{\partial \sigma} = SN'(d_1)\sqrt{T-t}
\end{aligned}
$$

   Show that the call satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0.$$

   HINT! First, show that

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

……………………………………………………………………………………(10p)

5. The price of the stock of ABC corporation satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

   where W_t is a Brownian motion. The corporation enters into a contract with its CEO, worth

$$A\ln\left(\frac{S_T}{K}\right)$$

at time T. Note that if the stock price S_T is greater than K, the CEO receives a payment, but if $S_T < K$ then she has to pay the corporation. In other words, this is an incentive for her to see that the stock price goes up. In order to neutralize the contract, she decides to hedge. Ignoring transaction costs, how much does it cost her at time $t = 0$ to implement a hedge that will exactly balance this contract at time $t = T$? You should obtain your answer by

(a) Expressing the hedging cost in terms of risk neutral expectations,

(b) evaluating these expectations.

(c) Finally, work out an actual cost, where T corresponds to 2 years, $r = 3\%$ per year, $\mu = 6\%$ per year, $\sigma = 30\%$ per year, $K = 10$, the initial price of the stock is $S_0 = 12$, and $A = 100{,}000$.
....(10p)

6. Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

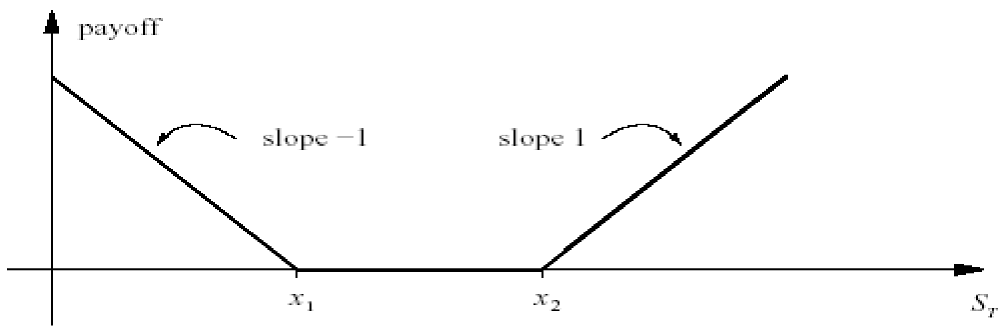
$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

(a) Suppose that you for some reason are fairly certain that there will be a large move in the stock price until time T. However you are not certain of whether the price will increase or decrease. One way to make use of your information is to buy a strangle, which is a T-contract with a payoff structure illustrated in the figure below.



(For the application described above today's stock price should lie between x_1 and x_2.) Compute the price of the strangle as explicitly as possible.

Hint: The easiest way of doing this might be to construct a portfolio of derivatives with known prices, which at time T will pay exactly the same amount as the option above, i.e. to use the same method used to derive put-call-parity..(4p)

(b) Determine the arbitrage price of the contingent T-claim $X = \Phi(S_T)$ with contract function Φ given by

$$\Phi(s) = \begin{cases} \sqrt{s} & \text{if } s > K \\ 0 & \text{otherwise.} \end{cases}$$

Here K denotes a strictly positive constant.
………………………………………………………………………………………(6p)

7. Consider a model for two countries. We then have a domestic market (Sweden) and a foreign market (Japan). The domestic and foreign interest rates, r_d and r_f, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = e^{r_d t} \quad B_t^f = e^{r_f t}$$

where B^d and B^f are denominated in units of domestic and foreign currency, respectively. The exchange rate process X, which is used to convert foreign payoffs into domestic currency (the "krona/yen"-rate), is modeled by the following stochastic differential equation under the objective measure P

$$dX = \mu_X X dt + \sigma_X X dW$$

where μ_x and σ_x are assumed to be constants and W is a P-Wiener process. A domestic martingale measure, Q^d, is a measure which is equivalent to the objective Measure P and which makes all a priori given price process, expressed in units of domestic currency and discounted using the domestic risk-free rate, martingales. We assume that if you buy the foreign currency this is immediately invested in a foreign bank account. All markets are assumed to be frictionless.

(a) Determine the Q^d-dynamics of X.

(b) Now take the viewpoint of a foreign-based investor, that is an investor who consistently denominates her profits and losses in units of foreign currency. A foreign martingale measure, Q^f, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of foreign currency and discounted using the foreign risk-free rate, martingales.

Find the Girsanov transformation between Q^d and Q^f.

(c) The domestic (foreign) market is said to be risk neutral if the domestic (foreign) martingale measure is equal to the objective measure P. Under which conditions are both markets risk neutral?
………………………………………………………………………………………(10p)

8. <u>Extra problem(see page 1)</u>: Derive Black-Scholes Partial Differential Equation and solve it for a European Put-option.

Formulas:

- Suppose that there exist processes $X(\cdot, T)$ for every $T \geq 0$ and suppose that Y is a process defined by:

  $$Y(t) = \int_t^{T_0} X(t,s)ds$$

  Then we have the following version of Itô's formula

  $$dY_t = -X(t,t)dt + \int_t^{T_0} dX(t,s)ds$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

  $$F(t,s) = S \cdot N\left[d_1(t,S)\right] - e^{-r(T-t)} K \cdot N\left[d_2(t,S)\right]$$

  Here N is the cumulative distribution function for the $N(0, 1)$ distribution and

  $$d_1(t,S) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\},$$
  $$d_2(t,S) = d_1(t,S) - \sigma\sqrt{T-t}$$

- If N denotes the cumulative distribution function for the $N(0; 1)$ distribution, then

  $N(-x) = 1 - N(x)$.

A linear SDE of the form

$$\begin{cases} dX_t = \left(aX_t + b_t\right)dt + \sigma_t dW_t \\ X_0 = x_0 \end{cases}$$

where a is a constant and b_t and σ_t are deterministic functions, has the solution

$$X_t = e^{at}x_0 + \int_0^t e^{a(t-s)} b_s ds + \int_0^t e^{a(t-s)} \sigma_s dW_s$$

1.4 Normal Distribution Table

x	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Probability that a normal random variable is smaller than x.