

Examination in MMA707, Analytical Finance I

Wednesday 3 of November 2010, 14:10 – 18:30

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You may use: Calculator (without alphanumeric capability), pencil, ruler and rubber gum.

General direction: The solution should be well motivated and readable. All notations must

be explained.

Good Luck!!

1. (a) Compute the price of a European call option with strike price *K* = 125 and
 exercise time *T* = 2 years, using a binomial tree with two trading dates *t1* = 0
 and *t2* = 1 (your portfolio at time *t3* = 2 is the same as your portfolio at time
 *t2* = 1) and parameters *s0* = 100, *u* = 1.5, *d* = 0.5, *r* = 0, and *p* = 0.75

(b) i. Consider a discrete time financial model with dates *t* = 0, 1,…, *T*, a risk free
 asset with price process *B*, and a stock with price process *S*, i.e.

 *B*(*t*) = price at time *t* of the risk free asset;
 *S*(*t*) = price at time *t* of the stock.

 Now let *ht* = (*xt, yt*) denote the portfolio which is held from *t* - 1 until *t*, i.e.

 *xt* = number of risk free assets held in the time interval (*t – 1, t*]
 *yt* = number of stocks held in the time interval (*t – 1, t*]

 What does it mean that a portfolio is self-financing in this model?

 ii. Consider a standard Black-Scholes market, i.e. a market consisting of a risk
 free asset, *B*, with *P*-dynamics given by:

  and a stock, *S*, with *P*-dynamics given by

 

 Here *W* denotes a *P*-Wiener process and *r*, *α* and *σ* are assumed to be
 constants. Now let *ht* = (*h0t, h1t*) denote the portfolio held at time *t* in this
 model, i.e.

 *h0t* = number of risk free assets held at time *t*
 *h1t* = number of stocks held at time *t.*

 What does the self-financing condition look like for this model?

(c) Consider the standard Black-Scholes model described in (b). For a given
 portfolio *h* the **relative portfolio** *u* = (*u0*, *u1*) is given by

 

 at time *t*. Here *Vh* denotes the value process associated with the portfolio *h*.
 Note that *u0t + u1t* = 1.

 i. What does the self-financing condition look like in terms of the relative
 portfolio?
 ii. Regard the constant relative portfolio *u* = (½, ½) as self-financing and
 determine the value process associated with it, given that the initial wealth
 invested in it is *V0 = v*.

(d) Use the binomial tree in (a) to find a replicating portfolio for the option in (a)
 and verify that the portfolio is self-financing
…...……..………………………………………………….………………….(10p)
2. Below is a picture of a one-period (time points *t* = 0 and *t* = 1) binomial model with parameters *S0* = 100, *u* = 1.5, *d* = 0.5 and *p* = 0.75.
 
(a) What are the arbitrage bounds for the interest rate *r*?
(b) Given that the price at time *t* = 0 of a European call option with strike price
 *K* = 108 kr and exercise time *T* = 1 year has been computed to 22 kr, what is the
 interest rate *r*?
……..………..............…………………………………………………..………….(5p)
3. You are going to buy a European derivative in the Black-Scholes world. From the trading software you get the following data:

 Underlying price = 100.0
 Risk-free interest rate = 6.0%
 Option Delta = 0.597866
 Option Gamma = 0.013659
 Option Theta = -13.76591
 Underlying Volatility = 40.0%

Calculate the price of the derivative.
….............…………………………………………………………………………..(5p)
4. Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, *B*, with *P*-dynamics given by:

 and a stock, *S*, with *P*-dynamics given by

 

Here *W* denotes a *P*-Wiener process and *r*, *α* and *σ* are assumed to be constants.
(a) Derive the put-call parity in this model.
(b) Define the concepts portfolio, value process, relative portfolio and self-financing
 portfolio.
(c) Consider the following relative portfolio

 

Relative portfolios can always be interpreted as relative portfolios of self-financing portfolio strategies. Given that the initial value of the portfolio should be *V0*, which self-financing portfolio strategy does the above relative portfolio correspond to, and what does the value process for this portfolio look like?

(c) The broker firm F&H has introduced the derivative "the inverse mean" on the
 market. This contract is specified by two fixed points in time *T0* and *T1*, with
 *T0* < *T1* . The holder of this contract obtains the sum

 

 at time *T1*. Determine the price process П(*t, X*) for *t* < *T0*.
………...…………………………………………………………….…………….(10p)
5. Consider a two-dimensional Black-Scholes market, i.e. a market consisting of a risk free asset, *B*, with *P*-dynamics given by:

 and two stocks, *X* and *Y* with *P*-dynamics given by

 

Here *V* and *W* denotes two independent *P*-Wiener processes and *r*, *α, β, ρ, γ* and *σ* are assumed to be constants.

Assume that the filtration is the natural filtration generated by the Wiener processes *W* and *V*. Show that this model is free of arbitrage and complete given that *γσ ≠*  0.
……...……………………………………………………….……………….(10p)
6. a) Prove that the Black-Scholes model is complete, i.e. that all contingent claims are
 reachable.
b) Prove that the Black-Scholes model is free of arbitrage
…………………………………….…………………………….……………….(10p)

**Formulas:**

* Suppose that there exist processes *X*(**.**, *T*) for every T ≥ 0 and suppose that *Y* is a process defined by:

 

Then we have the following version of Itô's formula

 
* The standard Black-Scholes formula for the price П(*t*) of a European call option with strike price *K* and time of maturity *T* is П (*t*) = *F*(*t*, *S*(*t*)), where

 

Here *N* is the cumulative distribution function for the *N*(0, 1) distribution and

 
* If *N* denotes the cumulative distribution function for the *N*(0; 1) distribution, then

 *N*(-*x*) = 1 - *N*(*x*).

A linear SDE of the form

 

where *a* is a constant and *bt* and *σt* are deterministic functions, has the solution

 

