Analytical Finance – Problems and solutions

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In all problems we use the following notations if nothing else is said:

$B(t)$	The value of the money market account at time t
r	The risk-free interest rate
R	A short notation of $1 + r$
Ω	A sample space
ω_i	Outcome i from a sample space Ω
$S(t)$	Price of a security (financial instrument, equity, stock) at time t
$F(t)$	The forward price of a security (financial instrument, equity, stock) at time t
q	The risk neutral (risk-free) probability of an increase in price
p	The objective (real) probability or the risk-free probability of an decreasing price
Q	The risk neutral probability measure.
P	The objective (real) probability measure.
$E^Q[\,\cdot\,]$	The expectation value with respect to Q
$Var^Q[\,\cdot\,]$	The variance with respect to Q
ρ	The risk premium
$X(t)$	A stochastic value/process
I_t	The information set at time t
u	The binomial "up" factor with risk neutral probability p_u or q.
d	The binomial "down" factor with risk neutral probability p_d or p.
Z	A stochastic variable
$V(t)$	A value (process)
μ, α	The drift in a stochastic process
σ	The volatility in a stochastic process
t	Time
T	Time to maturity
K	The option Strike Price
λ	The market price of (volatility) risk (the sharp ratio)
C	A (call) option value
Δ	The change in the option value w.r.t. the underlying price, S
Γ	The change in the option Δ w.r.t. the underlying price, S
ν	The change in the option value w.r.t. the volatility, σ
Θ	The change in the option value w.r.t. time, t
ρ	The change in the option value w.r.t. the interest rate, r
d_1, d_2	Coefficients (variables) in the Black-Scholes model
VaR	Value-at-Risk
F	A set or subsets to the sample space Ω
μ	A finite measure on a measurable space
$W(t)$	A Wiener process
$N[\mu, \sigma]$	A Normal distribution with mean μ and variance σ

τ	A stopping time (usually for American options)
L_t	A likelihood function of time t

Problem 1

In Cox-Ross-Rubenstein binomial model the underlying price might increase by a factor $u = e^{\sigma\sqrt{\Delta t}}$ or decrease with a factor $d = 1/u = e^{-\sigma\sqrt{\Delta t}}$. Derive the risk neutral probabilities for the model it the continuous risk-free interest is given by r.

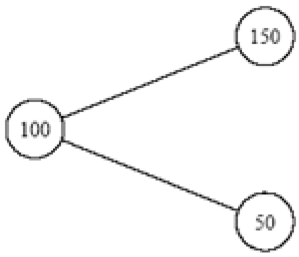
Problem 2

Calculate the price of a European call option with strike price $K = 125$ and maturity time $T = 2$ years, using a binomial tree with two trading dates $t_1 = 0$ and $t_2 = 1$ years. The initial stock price ($S(0)$) is 100 and the price might increase with a factor $u = 1.5$ or decre6ase with a factor $d = 0.5$ in each time-step. The risk-free interest rate (r) is 0, and the objective probability that the price will increase is $p = 0.75$.

Use the binomial tree in to find a replicating portfolio for the option in and verify that the portfolio is self-financing.

Problem 3

Below is a picture of a one-period binomial model at time $t = 0$ and $t = 1$ with a initial stock price $S(0) = 100$ and parameters $u = 1.5$, $d = 0.5$. The objective probability that the price will increase is $p = 0.75$.



What are the arbitrage bounds for the interest rate r?

Given the binomial tree (and data) above and that the price at time $t = 0$ of a European call option with strike price $K = 108$ and exercise time $T = 1$ year has been computed to 22, what is the interest rate r?

Problem 4

Calculate the price of a European call option with strike price K = 100 and maturity date T = 2 years, using a binomial tree with two trading dates t_1 = 0 and t_2 = 1. The initial stock price (S(0)) is 100 and the price might increase with a factor u = 1.5 or decre6ase with a factor d = 0.5 in each time-step. The risk-free interest rate (r) is 10%, and the objective probability that the price will increase is p = 0.75.

Problem 5

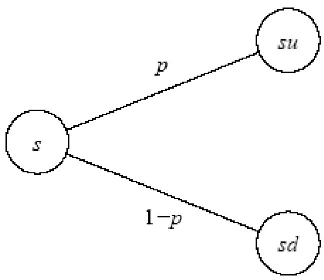
Calculate the price of an American put option with strike price K = 90 and exercise time T = 2 years, using a binomial tree with two trading dates t_1 = 0 and t_2 = 1. Your portfolio at time t_3 = 2 is the same as your portfolio at time t_2 = 1 and parameters given by S(0) = 100, u = 1.5, d = 0.5, r = 0, and p = 0.75.

Problem 6

Compute the price of an American put option with strike price K = 100 and exercise date T = 2 years, using a binomial tree with two trading dates t_1 = 0 and t_2 = 1. Your portfolio at time t_3 = 2 is the same as at time t_2 = 1 and parameters given by S(0) = 100, u = 1.4, d = 0.8, r = 10%, and p = 0.75.

Problem 7

Show that the one period binomial model is complete given that $d < 1+r < u$.



Problem 8

Consider a discrete time financial model with dates $t = 0, 1,\dots, T$, a risk free asset with price process B, and a stock with price process S, i.e.

> $B(t)$ = price at time t of the risk free asset;
> $S(t)$ = price at time t of the stock.

Now let $h_t = (x_t, y_t)$ denote the portfolio which is held from t - 1 until t, i.e.

> x_t = number of risk free assets held in the time interval $(t - 1, t]$
> y_t = number of stocks held in the time interval $(t - 1, t]$

What does it mean that a portfolio is self-financing in this model?

Problem 9

For each of the following four statements indicate whether it is true or false in a standard Black-Scholes setting. It is also possible not to answer at all. A correct answer will be rewarded with one point, an incorrect answer results in minus one point, and no answer gives zero points. If the total sum of points is negative you will get zero points.

i. Implied volatility is estimated from historical data on prices of the underlying asset.
ii. The delta of a contingent T-claim X is the rate of change of its price with respect to the price of the underlying asset.
iii. Assume that $h = (h_1,\dots, h_n)$ and $S = (S_1,\dots, S_n)$ are adapted processes with stochastic differentials representing a portfolio process and a price process. If

$$V = \sum_{i=1}^{n} h_i S_i \ ,$$

then $dV = hdS$.
iv. The local rate of return of an ideally traded asset is always equal to the short rate under the risk-neutral martingale measure Q.

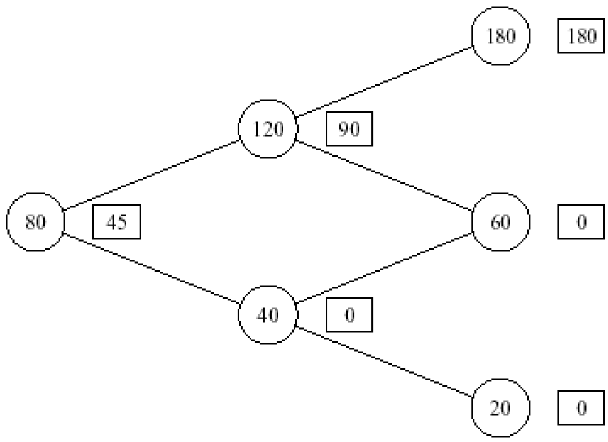
Problem 10

In the binomial tree below the price of a binary asset-or-nothing option with expiry in two years and payoff

$$X = \begin{cases} S(2) & \text{if } S(2) > 120 \\ 0 & \text{otherwise} \end{cases}$$

has been calculated using the parameters $S(0) = 80$, $u = 1.5$, $d = 0.5$, $r = 0$, and $p = 0.50$. Here $S(t)$ denotes the stock price at time t, u and d the binomial factors for price up and down, r the risk-free interest rate and p the objective probability for an increasing price of factor u.

In the definition of the contract function $S(2)$ denotes the stock price at time $t = 2$. The value of the stock is written in the nodes below, and the value of the option in the adjacent boxes.



Find the replicated portfolio for this option and verify that the option is self-financing.

Problem 11

Show that, under the risk neutral measure Q: (p, q), the discounted stock price $\left\{(1+r)^{-k} S_k, \mathcal{F}_k\right\}_{k=0}^{n}$ in the binomial model is martingale.

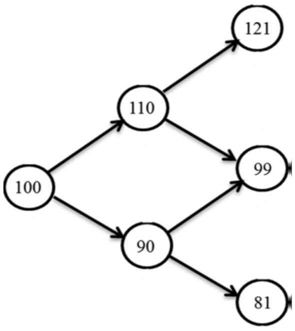
Problem 12

(a) Suppose that a share price S is currently \$100, and that tomorrow it will be either \$101, with probability p, or \$99, with probability $1 - p$. A call option, with value C, has exercise price \$100. Set up a Black-Scholes hedged portfolio and hence find the value of C. (Ignore interest rates.)

(b) Now repeat the calculation for a cash-or-nothing call option with payoff \$100 if the final asset price is above \$100, zero otherwise. What difference do you notice? This very simple discrete model is the basis of the binomial method.

Problem 13

Consider a two period binomial model where the stock price evolves according to the figure bellow and the probability of an up-move is $p = 0.5$ under the original, real P-measure. The interest rate in each period is constant equal to $r = 2\%$.



a.) Compute the price of a European put option with the strike price $K = 105$.
b.) Compute the price of an American put option with the strike price $K = 105$.
c.) Compute the price of an up-and-out put barrier option with barrier $L = 105$ and strike $K = 105$ and discuss the results when comparing with a.) and b.).
d.) Compute the result of a European lookback option with strike $K = 105$.

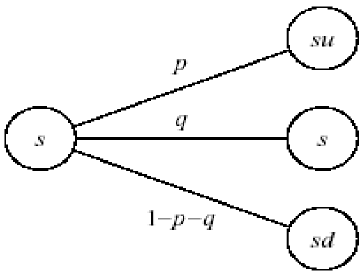
Problem 14

Consider the binomial method with $S_u = uS$, $S_d = S/u$, $u = e^{\sigma\sqrt{\delta t}}$ where δt is the time step. Expand the recurrence relation to $O(\delta t)$ (use Taylor series about the point (S, t)) to derive the Black-Scholes equation in the limit $\delta t \to 0$.

HINT: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ...$

Problem 15

Consider a one period model very similar to the one period binomial model, the only difference being that the stock price S can also stay the same with a certain probability q, as depicted in the figure below.



Given that $d < 1 + r < u$ is the model complete or not? A proof or counterexample is required.

Problem 16

Derive the Itô formula (Itô lemma).

Problem 17

Calculate the integral $\int_0^t W(s)dW(s)$ where W is a Wiener process.

Problem 18

In the standard Black-Scholes model the stock price S is assumed to follow a geometric Brownian motion

$$\begin{cases} dS_u = \alpha S_u du + \sigma S_u dW_u \\ S_t = s \end{cases}$$

Here W denotes a Wiener process where α and σ are constants.

(a) Derive an explicit solution of the stochastic differential equation.
(b) Determine the distribution of the solution to the stochastic differential equation.

Problem 19

Solve the following stochastic differential equation for the Likelihood process

$$\begin{cases} dL(t) = g(t)L(t)dW(t) \\ L(0) = 1 \end{cases}$$

Here W denotes a Wiener process where α and σ are constants.

Problem 20

Solve the following stochastic differential equation for the Likelihood process

$$\begin{cases} dX(t) = \mu dt + \sigma dW(t) \\ X(0) = x \end{cases}$$

Here W denotes a Wiener process where α and σ are constants.

Problem 21

Solve the following stochastic differential equation

$$\begin{cases} dX(t) = \mu X(t)dt + \sigma dW(t) \\ X(0) = x \end{cases}$$

where W denotes a Wiener process where μ and σ are constants.

Problem 22

Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

i. Check whether the portfolio defined by $h(t) = (h^B(t), h^S(t)) = (S(t), B(t))$ is self-financing or not.
ii. Determine whether the following process X represents a tradable asset or not, where $X_t = S_t^{-\beta}$, where $\beta = 2r/\sigma^2$.

Problem 23

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants. Check whether the portfolio defined by

$$h_t = \left(-\frac{S_t^2}{2}, e^{rt} S_t \right)$$

is self-financing or not.

Problem 24

A security $S(t)$ is supposed to follow a process

$$\frac{dS}{S} = \mu dt + \sigma dW(t), \quad W(0) = 0$$

where W is a normalized Wiener process (under the real probabilities) and where t is given in years. The drift and volatility are given by $\mu = 0.12$ and $\sigma = 0.30$ and the risk-free interest rate $r = 6\%$. Calculate the arbitrage-free price on a contract with a payout two years from now, given by:

$$S(0) \cdot \frac{S(2)}{S(1)}.$$

Problem 25

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants. Derive the put-call parity in this model.

Problem 26

Prove the put-call relationship for American options (current time is $t = 0$ maturity at T)

$$S - K \leq C_A - P_A \leq S - Ke^{-rT}$$

Problem 27

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants. Now let $h_t = (h^0_t, h^1_t)$ denote the portfolio held at time t in this model, i.e. h^0_t is number of risk free assets held at time t. h^1_t the number of stocks held at time t. What does the self-financing condition look like for this model?

Problem 28

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants. For a given portfolio h the **relative portfolio** $u = (u^0, u^1)$ is given by

$$u_t^0 = \frac{h_t^0 B(t)}{V^h(t)}, \qquad u_t^1 = \frac{h_t^1 S(t)}{V^h(t)}$$

at time t. Here V^h denotes the value process associated with the portfolio h. Note that $u^0_t + u^1_t = 1$.

i. What does the self-financing condition look like in terms of the relative portfolio?
ii. Regard the constant relative portfolio $u = (½, ½)$ as self-financing and determine the value process associated with it, given that the initial wealth invested in it is $V_0 = v$.

Problem 29

Find a solution to the following heat equation

$$\begin{cases} F_t + \frac{1}{2}\sigma^2 F_{xx} = 0 \\ F(T,x) = x^2 \end{cases}$$

Problem 30

Solve the following partial differential equation

$$\begin{cases} \frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0 \\ F(T,x) = x^2 \end{cases}$$

Problem 31

Solve the following partial differential equation

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 F}{\partial x^2}(t,x) = rF(t,x) \\ F(T,x) = x^2 \end{cases}$$

Problem 32

Suppose an underlying security follows a normal distributed process where the current future price, strike price, risk free interest rate, volatility, and time to maturity is denoted as f, K, r, σ, and $T - t$ respectively. The current future price then follows the following normal distributed process

$$df = \mu dt + \sigma dW_t$$

where μ is a constant drift. For instruments like swaptions, f represents the forward rate. In a risk neutral world the process is giving as

$$df = \sigma dV_t$$

with the trivial solution, from integration over the interval $[t, T]$:

$$f(T) = f(t) + \sigma(V_T - V_t)$$

We see that f is a Gaussian process; $N[f_t, \sigma^2(T - t)]$, i.e., with mean $f(t)$ and variance $\sigma^2(T - t)$. Derive the price of a call option on a future were we assume the price (forward rate) is normal distributed.

Problem 33

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

(a) Determine the arbitrage free price of an option which at time T pays either K or the value of stock at time T, whichever the holder prefers. The contract function describing the option is thus given by

$$\Phi(T) = \max\{S_T, K\}.$$

Hint: The easiest way of doing this might be to construct a portfolio of derivatives with known prices, which at time T will pay exactly the same amount as the option above, i.e. to use the same method used to derive put-call-parity.

(b) Now consider an option which has the same contract function as the option in (a), except for that K is replaced by $S_{T(0)}$, where $T(0)$ is a fixed time such as $T(0) < T$. Determine the arbitrage price of this option for $t \in [0, T(0)]$.

Problem 34

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants. Determine the arbitrage price at time t for $t \in [0, T]$ of a T-claim defined by

$$X = \Phi(S_T) = \sqrt{S_T}$$

Problem 35

It is well-known that the price process Π of a simple contingent claim $X = \Phi(S_T)$ is given by $\Pi(t; X) = F(t, S_t)$ where F is the solution to Black-Scholes equation

$$\begin{cases} \frac{\partial F}{\partial t} + r \cdot s \cdot \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \cdot \frac{\partial^2 F}{\partial s^2} - r \cdot F = 0 \\ F(T, s) = \Phi(s) \end{cases}$$

Use this characterization of the price to find explicit expressions for the replicating portfolio of a simple contingent claim (the expressions may contain the pricing function F and derivatives thereof).

Problem 36

Calculate the Greeks to the Black-Scholes model, i.e., delta, gamma, rho, vega and theta.

Problem 37

From the result of problem 35, determine the self-financing portfolio associated with the value process

$$V(t) = S^2(t)\exp\left\{(r + \sigma^2)(T - t)\right\}$$

Problem 38

Consider a financial market model with a risk-free asset B, with the same dynamics as in the Black-Scholes model, and two stocks S^1 and S^2, with the following dynamics under the objective measure P

$$\begin{cases} dS_t^1 = \alpha_t^1 S_t^1 dt + \sigma_t^1 S_t^1 dW_t + \delta_t^1 S_t^1 dN_t^1 \\ S_0^1 = s^1 \end{cases}$$

$$\begin{cases} dS_t^2 = \alpha_t^2 S_t^2 dt + \sigma_t^2 S_t^2 dW_t + \delta_t^2 S_t^2 dN_t^2 \\ S_0^2 = s^2 \end{cases}$$

Here W is a P-Wiener process and N^1 and N^2 are Poisson processes with intensities λ_1 and λ_2, respectively (under P). Make an educated guess as to whether this model is arbitrage free and/or complete. Motivate your answer and if you believe that the model is not free of arbitrage or incomplete suggest a suitable modification of the model which will remedy this.

Problem 39

Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants. Determine the arbitrage price of the contingent T-claim $X = \Phi(S_T)$ with contract function Φ given by

$$\Phi(s) = \begin{cases} \sqrt{s} & \text{if } s > K \\ 0 & \text{otherwise.} \end{cases}$$

Here K denotes a strictly positive constant.

Problem 40

Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here *W* denotes a *P*-Wiener process and *r*, α and σ are assumed to be constants. Determine the arbitrage price of the contingent *T*-claim $X = \Phi(S_T)$ with contract function Φ given by

$$\Phi(s) = \begin{cases} s^2 & \text{if } s > K \\ 0 & \text{else} \end{cases}$$

Problem 41

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, *B*, with *P*-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, *S*, with *P*-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here *W* denotes a *P*-Wiener process and *r*, α and σ are assumed to be constants. Derive the put-call parity in this model.

(a) Define the concepts portfolio, value process, relative portfolio and self-financing portfolio.

(b) Consider the following relative portfolio

$$u = \left(u^B, u^S\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Relative portfolios can always be interpreted as relative portfolios of self-financing portfolio strategies. Given that the initial value of the portfolio should be V_0, which self-financing portfolio strategy does the above relative portfolio correspond to, and what does the value process for this portfolio look like?

Problem 42

The price of the stock of ABC corporation satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a Brownian motion. The corporation enters into a contract with its CEO, worth

$$A \ln\left(\frac{S_T}{K}\right)$$

at time T. Note that if the stock price S_T is greater than K, the CEO receives a payment, but if $S_T < K$ then she has to pay the corporation. In other words, this is an incentive for her to see that the stock price goes up. In order to neutralize the contract, she decides to hedge. Ignoring transaction costs, how much does it cost her at time $t = 0$ to implement a hedge that will exactly balance this contract at time $t = T$? You should obtain your answer by

(a) Expressing the hedging cost in terms of risk neutral expectations,
(b) Evaluate these expectations.
(c) Finally, work out an actual cost, where T corresponds to 2 years, r = 3% per year, μ = 6% per year, σ = 30% per year, K = 10, the initial price of the stock is S_0 = 12, and A = 100,000.

Problem 43

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here *W* denotes a *P*-Wiener process and *r*, α and σ are assumed to be constants. By a digital or binary option we mean a contract whose payoff depends in a discontinuous way on the terminal price of the underlying asset. The simplest examples of binary options are cash-or-nothing options and asset-or-nothing options. The payoffs at maturity of a cash-or-nothing call and a cash-or-nothing put option are, respectively

$$\begin{cases} BCC(T) = K \cdot I_{\{s(T)>K\}} \\ BCP(T) = K \cdot I_{\{s(T)<K\}} \end{cases}$$

where in both cases *K* denotes a pre specified amount of cash. Similarly, for the asset-or-nothing options we have

$$\begin{cases} BAC(T) = S(T) \cdot I_{\{s(T)>K\}} \\ BAP(T) = S(T) \cdot I_{\{s(T)<K\}} \end{cases}$$

for a call and put, respectively. Your task is to determine the arbitrage free price of a binary asset-or-nothing call and a cash-or-nothing call.

Problem 44

You are going to buy a European derivative in the Black-Scholes world. From the trading software you get the following data:

Underlying price = 100.0
Risk-free interest rate = 6.0%
Option Delta = 0.597866
Option Gamma = 0.013659
Option Theta = -13.76591
Underlying Volatility = 40.0%

Calculate the price of the derivative.

Problem 45

Calculate the price in EUR on a European option to buy 1 000 USD for 1 100 EUR one year from now. The price of a USD today is 0.98 EUR. The annual interest rate in EUR is 3 % and in USD 5 % continuous compounded. The time dependent volatility of the USD price (in EUR) is supposed to be 0.25/(3 + *t*) from *t* = 0 to *t* = 1 (*t* = time in years).

Problem 46

Calculate the price in EUR on an American option to buy 1 000 USD for 990 EUR one year from now. The price of a USD today is 0.98 EUR. The annual interest rate in EUR is 3% and in USD 5% continuous compounded. The volatility of the USD price (in EUR) is supposed to be 6%.

Problem 47

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, *B*, with *P*-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, *S*, with *P*-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here *W* denotes a *P*-Wiener process and *r*, α and σ are assumed to be constants. Suppose that the following parameters have been estimated for the above model: $\alpha = 0.08$, $\sigma = 0.2$, and $r = 0.05$. Furthermore suppose that at time $t = 0$ you sell a European call option on the stock with strike price $K = 108$ and exercise date $T = 1$ year. Your task is to evaluate the performance of a delta hedge for this short position in the option. More precisely, if you sell the option and hedge the position using delta hedging as described below, will you have lost or gained money at time *t* = l?

The hedge should be rebalanced every four months (i.e. at *t* = 1/3 year and so on). Furthermore, the hedge should be set up at time *t* = 0 in such a way that the value of the total portfolio is zero, and the rebalancing should be performed in a self-financing manner. Below you find the necessary data to compute the hedge.

Time	0	1/3	2/3	1
Stock price	100	114	104	108
Option price	6.78	12.87	3.81	0
Δ of option	0.4861	0.7313	0.4504	-

Make sure to explain how you obtain your numbers!

Problem 48

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, *B*, with *P*-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, *S*, with *P*-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here *W* denotes a *P*-Wiener process and *r*, α and σ are assumed to be constants. Determine the arbitrage-free price of a so called straddle, which is a contingent *T*-claim defined by $X = \Phi(S_T) = |S_T - K|$. A straddle is thus a contract you buy if you believe that there would be large movements in the stock price, but you were not sure in which direction.

Hint: The easiest way of doing this might be to construct a portfolio consisting of derivatives with known prices, which at time *T* will pay exactly the same amount as the claim above, i.e. to use the same method used to derive put-call-parity.

Problem 49

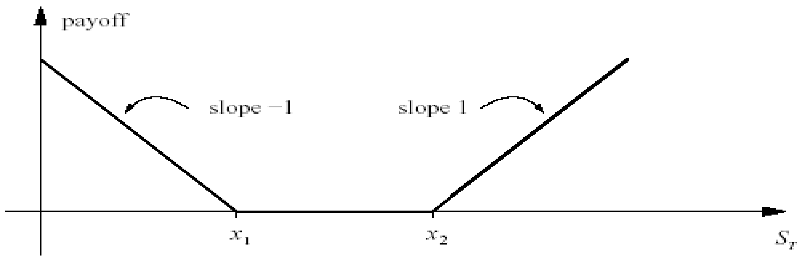
Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, *B*, with *P*-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, *S*, with *P*-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here *W* denotes a *P*-Wiener process and *r*, α and σ are assumed to be constants. Suppose that you for some reason are fairly certain that there will be a large move in the stock price until time *T*. However you are not certain of whether the price will increase or decrease. One way to make use of your information is to buy an strangle, which is a *T*-contract with a payoff structure illustrated in the figure below.



For the application described above today's stock price should lie between x_1 and x_2. Calculate the price of the strangle as explicitly as possible.

Problem 50

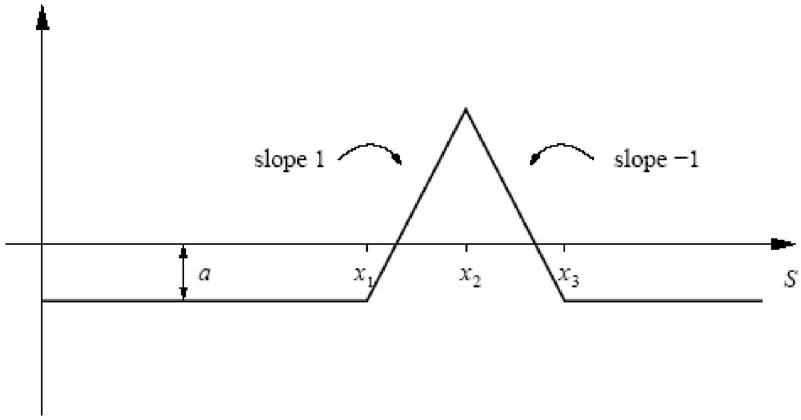
Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, *B*, with *P*-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, *S*, with *P*-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants. Suppose that you for some reason are fairly certain that the stock price will not move much until time T. In fact you are so certain of this you are willing to bet on it. Thus you would like to enter a buttery spread, which is a T-contract with the payout structure depicted in the figure below.



A natural choice for x_2 is today's stock price S_t and given that you have no information about whether a rise in the stock price is more likely than a fall you would set $x_2 = (x_1 + x_3)/2$. How narrow you want the interval $[x_1, x_3]$ depends on how sure you are that the stock price will not move. For this exercise set $x_1 = 0.95S_t$. Now suppose that you do not have much money at the moment, and therefore would not want to pay anything entering the contract. Determine how much you must be willing to lose at most, i.e. the constant a, if you do not want to pay anything for the contract today (at time t). The expression derived for a should be given only in terms of the parameters of the problem, but it may contain the density or distribution function for the $N(0, 1)$-distribution.

Problem 51

Consider a two-dimensional Black-Scholes market i.e. a market consisting of a risk-free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and two stocks, X and Y, with P-dynamics given by

$$\begin{cases} dX(t) = \alpha X(t)dt + \sigma X(t)dW(t) \\ dY(t) = \beta Y(t)dt + \delta Y(t)dW(t) \end{cases}$$

Here *W* is a one-dimensional *P*-Wiener process and *r*, *α*, *β*, *σ* and *δ* are assumed to be constants such as

$$r \neq \frac{\delta\alpha - \sigma\beta}{\delta - \sigma}$$

Assume that the filtration is the natural filtration generated by the Wiener process W. Show that this model is **not** free of arbitrage.

Problem 52

Consider a two-dimensional Black-Scholes market i.e. a market consisting of a risk-free asset, *B*, with *P*-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and two stocks, *X* and *Y*, with *P*-dynamics given by

$$\begin{cases} dX(t) = \alpha X(t)dt + \sigma X(t)dW(t) \\ dY(t) = \beta Y(t)dt + \delta Y(t)dW(t) \end{cases}$$

Here *W* and *V* are one-dimensional *P*-Wiener processes with correlation 1. *r*, *α*, *β*, *σ* and *δ* are assumed to be constants. Determine explicitly the Girsanov transformation between *P* and Q^X, where Q^X is the martingale measure for the numeraire process *X* (that is, under Q^X the process *B/X* and *Y/X* are martingales). Note that since there are two Wiener processes the likelihood process will be of the form

$$dL_t = g_t L_t dW_t + h_t L_t dV_t \,.$$

Your task is to determine *g* and *h*. Also give an explicit expression of the likelihood process.

Problem 53

Consider a model for two countries. We then have a domestic market (Sweden) and a foreign market (Japan). The domestic and foreign interest rates, r_d and r_f, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = e^{r_d t} \quad B_t^f = e^{r_f t}$$

where B^d and B^f are denominated in units of domestic and foreign currency, respectively. The exchange rate process X, which is used to convert foreign payoffs into domestic currency (the "krona/yen"-rate), is modeled by the following stochastic differential equation under the objective measure P

$$dX = \mu_X X dt + \sigma_X X dW$$

where μ_x and σ_x are assumed to be constants and W is a P-Wiener process.A domestic martingale measure, Q^d, is a measure which is equivalent to the objective Measure P and which makes all a priori given price process, expressed in units of domestic currency and discounted using the domestic risk-free rate, martingales. We assume that if you buy the foreign currency this is immediately invested in a foreign bank account. All markets are assumed to be frictionless.

(a) Determine the Q^d-dynamics of X.
(b) Now take the viewpoint of a foreign-based investor, that is an investor who consistently denominates her profits and losses in units of foreign currency. A foreign martingale measure, Q^f, is a measure which is equivalent to the objective measure P and which makes all a priori given price process, expressed in units of foreign currency and discounted using the foreign risk-free rate, martingales. Find the Girsanov transformation between Q^d and Q^f.
(c) The domestic (foreign) market is said to be risk neutral if the domestic (foreign) martingale measure is equal to the objective measure P. Under which conditions are both markets risk neutral?

Problem 54

Assume a one-period financial market model with three securities on the probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = P(\Omega)$ and $P(\omega_i) > 0$ $i = 1, 2, 3$. The current prices of the securities are $S(0) = (S_0(0), S_1(0), S_2(0)) = (100, 150, \alpha)$. At time $t = 1$ the prices are given by the following matrix:

$$S(1)=\begin{pmatrix} S_1(1,\omega_1) & S_1(1,\omega_2) & S_1(1,\omega_3) \\ S_2(1,\omega_1) & S_2(1,\omega_2) & S_4(1,\omega_3) \\ S_3(1,\omega_1) & S_3(1,\omega_2) & S_3(1,\omega_3) \end{pmatrix} = \begin{pmatrix} 110 & 110 & 110 \\ 154 & 198 & 143 \\ 176 & 220 & 143 \end{pmatrix}$$

a) Name an equivalent characterization to freedom of arbitrage in single period market models.

b) What are the possible values for α, so that the market remains arbitrage-free?

c) Assume that $\alpha = 160$. Calculate an equivalent martingale measure EMM with the bond as numéraire.

d) Calculate the price of the asset with payoff-vector $C(1) = (22, 66, 0)$

Problem 55

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants. The term Asian option is a generic name for a class of options whose terminal payoffs are based on average asset values during some period of the option's lifetime. Due to their averaging feature Asian options are suitable for assets which are not traded liquidly. Pricing Asian options is difficult. Your task is to determine the arbitrage price at time t for $t \in [0, T_1]$ of the simpler T_1-claim defined by

$$X = \frac{1}{\Delta T}\int_{T_0}^{T_1} S(u)du$$

Here $T_1 > T_0 \geq 0$ are two fixed times and $\Delta T = T_1 - T_0$. The claim is thus averaging, but there is no option feature.

Problem 56

Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, *B*, with *P*-dynamics given by

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

and a stock, *S*, with *P*-dynamics given by

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

Here *W* denotes a *P*-Wiener process and *r*, α and σ are assumed to be constants. The broker firm F&H has introduced the derivative "the inverse mean" on the market. This contract is specified by two fixed points in time T_0 and T_1, with $T_0 < T_1$. The holder of this contract obtains the sum

$$X = \int_{T_0}^{T_1} \frac{1}{S_t} dt$$

at time T_1. Determine the price process $\Pi(t, X)$ for $t < T_0$.

Problem 57

Consider a given market consisting of one risky asset with price process *S*(*t*) and cumulative dividend process *D*(*t*) and a risk free asset *B*(*t*), with dynamics

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases}$$

where *r* denotes the (stochastic) short rate. Now consider a fixed contingent *T*-claim *X*. Define what is meant by a futures contract on *X* with time of delivery *T*. Derive a formula for the futures price process *F*(*t*, *T*, *X*) (you may assume that the filtration used to describe the information available on the market is generated by a Wiener process).

Problem 58

Consider a Black-Scholes model with a constant continuous dividend yield, i.e. *B*, *S* and *D* satisfy

$$\begin{cases} dB_t = rB_t dt \\ dS_t = \alpha S_t dt + \sigma S_t dW_t \\ dD_t = \delta S_t dt \end{cases}$$

where r, α, σ and δ are assumed to be constants and *W* denotes a *P*-Wiener process. Give an explicit formula for the futures price for the case when $X = S(T)$.

Problem 59

Let *W* be a standard Wiener process on $(Q, \mathcal{F}, P_0)$ where the filtration is the one generated by *W*. Fix a time interval $[0, T]$. Under the measure P_0, the process *X* has the dynamics; where σ is a known constant.

$$dX_t = \sigma\sqrt{X_t}\,dW_t$$

(a) Define, for each real number α, a Girsanov transformation such that the measure P_0 is transformed into a measure P_α, such that *X* under P_α solves the equation

$$dX = \alpha X dt + \sigma\sqrt{X}\,dW^\alpha$$

where W^α is a P_α-Wiener process. Your task is to give a precise description of this measure transformation, by specifying the dynamics of the corresponding likelihood process L^α, where

$$L_t^\alpha = \frac{dP_\alpha}{dP_0}, \quad \text{on } \mathcal{F}_t$$

(b) Determine, for every $t \leq T$, the maximum likelihood estimator $\hat{\alpha}(t)$ for the parameter α, based on observations of *X* over the interval $[0, t]$, i.e. the value of α that maximizes L^α. Note that the answer shall be expressed in terms of the process *X*, and simplified as far as possible.

Problem 60

(a) Let X and Y be any processes possessing stochastic differentials. We let

$$\int_0^t Y(s)dX(s)$$

denote the standard Itô integral. Now we define the Stratonovich Integral

$$\int_0^t Y(s) \quad dX(s)$$

by the following formula

$$\int_0^t Y(s) \quad dX(s) = \int_0^t Y(s)dX(s) + \frac{1}{2}\int_0^t \{dX(s) \cdot dY(s)\}$$

In the last integral we use the standard multiplication rules for products of differentials, i.e. $dWdt = 0$, $(dW)^2 = dt$, etc. Now assume that X has a differential of the form

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

and that the deterministic function $F(t, x)$ is continuously differentiable, once in the t-variable, and three times in the x-variable. The nice thing about the Stratonovich integral is that for this integral concept we have the standard form of the chain rule, as opposed to the Itô formula with the irritating second order term. More precisely the following hold

$$dF(t, X(t)) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X} \quad dX$$

Prove this formula!

(b) Let X be any process possessing a stochastic differential. use the result in (a) in order to compute, in terms of X, the Stratonovich integral

$$\int_0^t X(s) \quad dX(s)$$

Solution to problem 1

The factors in the binomial model is given by

$$u = e^{\sigma \cdot \sqrt{dt}}$$
$$d = e^{-\sigma \cdot \sqrt{dt}}$$

Where *dt* is the time interval between observations of the prices and σ the volatility of the underlying security. With continuous compounding of interest rate *r* we must have

$$S_0 = e^{-r \cdot dt} \left(q_u \cdot u \cdot S_0 + q_d \cdot d \cdot S_0 \right) = S_0 \cdot e^{-r \cdot dt} \left(q_u \cdot u \cdot + q_d \cdot d \right)$$

The sum of the risk-neutral probabilities must be 1 and therefore be given by

$$\begin{cases} q_u + q_d = 1 \\ q_u \cdot u + q_d \cdot d = e^{r \cdot dt} \end{cases}$$

then

$$q_u = \frac{1}{u-d} \cdot \left[e^{r \cdot dt} - d \right] \qquad q_d = 1 - q_u .$$

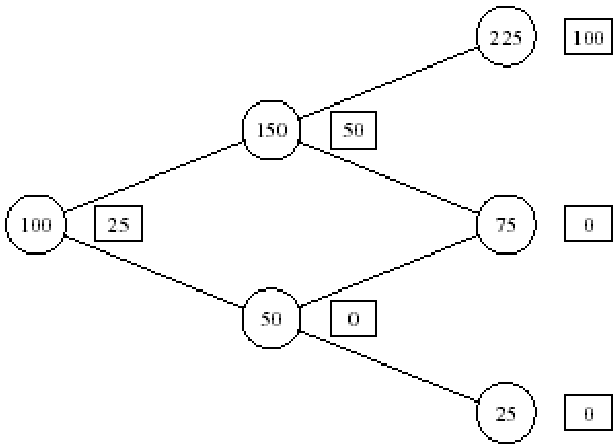
This means, that when we know the volatility, we also know the possible prices and the risk neutral probabilities. As we will see later, is that, also in the Black-Scholes model, as soon we know the volatility, we also know the price, since all other parameters are given. These parameters are the strike price, the time to maturity, the risk-free interest rate and the initial underlying price.

Solution to problem 2

First we have to calculate the risk-neutral (risk free, martingale) probabilities. For increasing price on the underlying stock we have

$$q = \frac{1 + r - d}{u - d} = \frac{1 - 0.5}{1.5 - 0.5} = 0.5$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option in adjacent boxes.



The price of the option is thus 25.

To obtain the replicating portfolio at t =0 we use the value process

$$V_t^h = x_t B_t + y_t S_t \ .$$

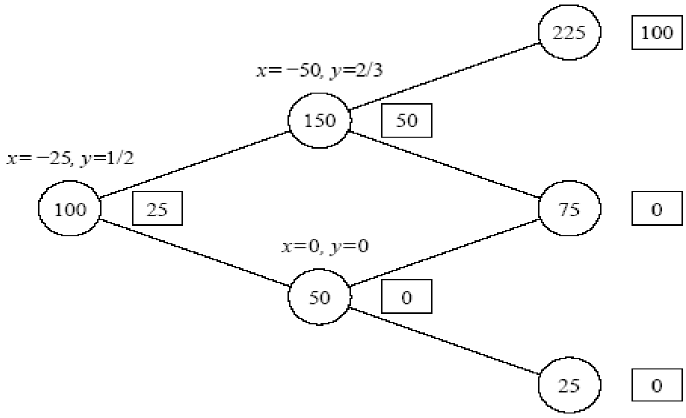
and solve the following set of equations

$$x + 150y = 50$$
$$x + 50y = 0$$

since regardless of whether the stock price goes up or down the value of the portfolio should equal the value of the option. This gives

$$x = -25, \ y = 1/2$$

Using the same method we find the rest of the replicating portfolio strategy and it is shown in the figure below.



That the portfolio strategy is self-financing is seen from the following equations

$$25 + \tfrac{1}{2} \times 150 = -50 + 2/3 \times 150$$
$$-25 + \tfrac{1}{2} \times 50 = 0 + 0 \times 50$$

Solution to problem 3

The arbitrage bound for the interest rate r is $d = 0.5 \leq (1 + r) \leq 1.5 = u$.

Both the price of stock and the price of the option have to satisfy the risk-neutral valuation principle. This gives us the following set of equations

$$\begin{cases} \frac{1}{1+r}\left[q \cdot 150 + (1-q) \cdot 50\right] = 100, \\ \frac{1}{1+r}\left[q \cdot 42 + (1-q) \cdot 0\right] = 22 \end{cases}$$

Solving these equations we find that r = 5% (and q = 0.55).

Solution to problem 4

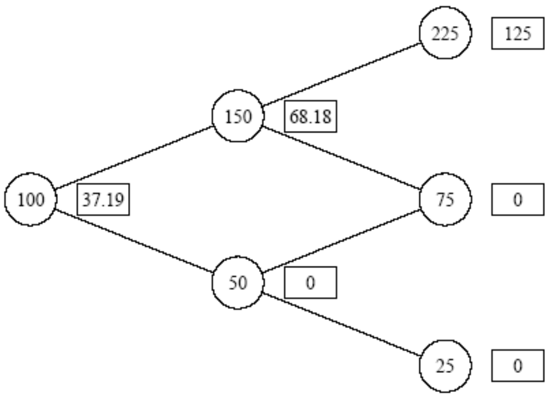
First we have to calculate the risk-neutral (risk free, martingale) probabilities. For increasing price on the underlying stock we have

$$q = \frac{1+r-d}{u-d} = \frac{1+0.1-0.5}{1.5-0.5} = 0.6$$

Using them we obtain the binomial tree below where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes. The value 68.18 adjacent to the node with stock price 150 is obtained as

$$\frac{1}{1.1}(0.6 \cdot 125 + 0.4 \cdot 0) = 68.18$$

and the other values are obtained analogously.



The price of the call option is thus

$$\frac{1}{1.1}(0.6 \cdot 68.18 + 0.4 \cdot 0) = 37.19$$

Solution to problem 5

First we have to calculate the risk-neutral (risk free, martingale) probabilities. For increasing price on the underlying stock we have

$$q = \frac{1+r-d}{u-d} = \frac{1-0.5}{1.5-0.5} = 0.5$$

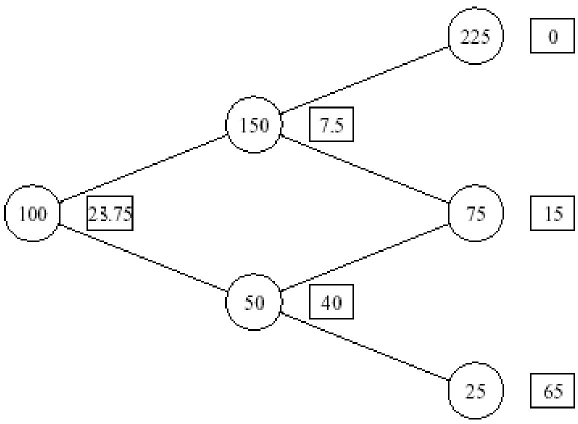
Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option in the adjacent boxes. The tree is created by multiplying with the up- and down factors, u = 1.5 and d = 0.5. The boundary conditions with strike price K = 90 are given by

max(90 – 225, 0) = 0, max(90 – 75, 0) = 15 and max(90 – 25, 0) = 65.

The "mid" values are calculated as

max(0.5 x 0 + 0.5 x 15, 90 – 150) = 7.5
max(0.5 x 15 + 0.5 x 65, 90 – 50) = 40

Since the interest rate r = 0 we have no discounting.



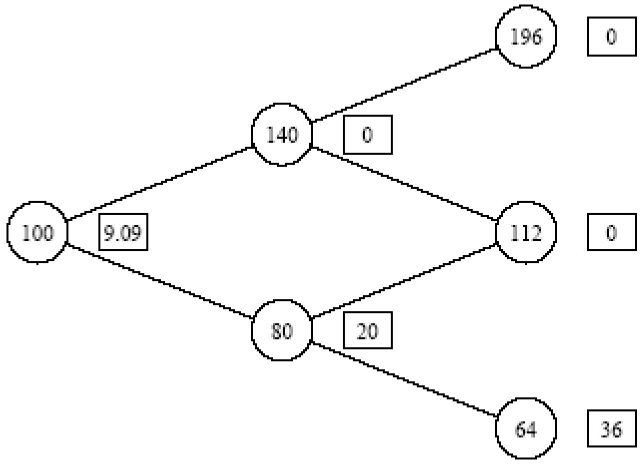
Finally we get the price of the option as (0.5 x 7.5 + 0.5 x 40) = 23.75.

Solution to problem 6

First we have to calculate the risk-neutral (risk free, martingale) probabilities. For increasing price on the underlying stock we have

$$q = \frac{1 + r - d}{u - d} = \frac{1 + 0.1 - 0.8}{1.4 - 0.8} = 0.5$$

Using them we obtain the binomial tree below where the value of the stock is written in the nodes, and the value of the option in the adjacent boxes.



The boundary conditions with strike price $K = 90$ are given by

max(100 – 196, 0) = 0, max(100 – 112, 0) = 15 and max(100 – 64, 0) = 36.

The lower "mid" values are calculated as

max{100 – 80, (1/1.1)·(0.5 x 0 + 0.5 x 36)} = 20

Thus, we have an early exercise of the option is optimal in this node! The price of the option is thus 20 x 0.5 / 1.1 = 9.09.

Solution to problem 7

Fix an arbitrary contingent claim $\Phi(Z)$, where Z denotes a random variable which equals u with probability p and d with probability $1 - p$. To show that the market is complete we have to show that we can find a replicating portfolio $h = \{x, y)$ for this contract. This means solving the following set of equations

$$x(1 + r) + ysu = \Phi(u)$$
$$x(1 + r) + ysd = \Phi(d)$$

(See the value process.) This is a system of linear equations in x and y, which has a unique solution if

$$\det\begin{bmatrix} 1+r & su \\ 1+r & sd \end{bmatrix} \neq 0$$

that is if $u \neq d$. Since $u > d$ this is the case, and the solution is given by

$$\begin{cases} x = \dfrac{1}{1+r}\dfrac{u \cdot \Phi(d) - d \cdot \Phi(u)}{u-d} \\ y = \dfrac{1}{s}\dfrac{\Phi(u) - \Phi(d)}{u-d} \end{cases}.$$

Since every contingent claim can be replicated the market is complete.

Solution to problem 8

The portfolio is self-financing if for t = 0, 1, 2,…., T - 1 we have

$$x_t B_t + y_t S_t = x_{t+1} B_t + y_{t+1} S_t$$

Solution to problem 9

i. False. Implied volatility is the volatility implied by an option price observed in the market.
ii. True.
iii. False. In order for the dynamics of the value process of a certain portfolio to have the stated form, the portfolio has to be self-financing.
iv. True.

Solution to problem 10

To obtain the replicating portfolio at t = 0 we have to use the value process

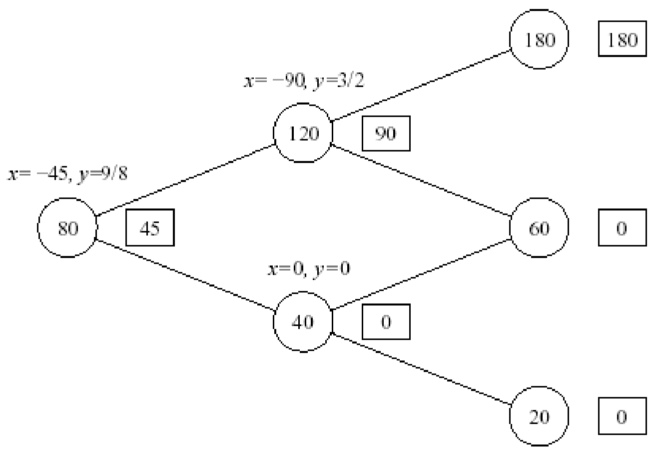
$$V_t^h = x_t B_t + y_t S_t \ .$$

Here x is the number of money we have in our money market account B and y the number of stocks, S. The value of x and y my take positive and negative values representing long and short positions. B_t is the value of one unit in the money market account at time t and S_t the value of the stock at time t. Therefore we solve the following system of equations

$$\begin{cases} x + y \cdot 120 = 90 \\ x + y \cdot 40 = 0 \end{cases}$$

The first equation represent the total portfolio value if the stock price reach the price of 120 at time $t = 1$ and the second equation if the stock price does fall to the value of 40. Due to the zero interest rate of the money market account the value of one unit in B is still 1. The value 90 is the value of our derivative and the value we want to replicate by B and S. I the stock price fall, the value of the derivative is 0.

The equations say that, regardless if the stock price goes up or down the value of the portfolio should equal the value of the option. When we solve this system of equations we get x = -45 and y = 9/8.With the same method we can replicate any node in the tree.



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At time t = 1 and if the stock price decrease to 180 we have to solve

$$\begin{cases} x + y \cdot 180 = 180 \\ x + y \cdot 60 = 0 \end{cases}$$

giving x = -90 and y = 3/2. If the stock price increase we have to solve:

$$\begin{cases} x + y \cdot 60 = 0 \\ x + y \cdot 20 = 0 \end{cases}$$

giving $x = y = 0$. As we see, we have to rebalance our portfolio regarding the stock goes up or down.

That the portfolio strategy is self-financing is seen from the following equations

$$\begin{cases} -45 + \dfrac{9}{8} \cdot 120 = -90 + \dfrac{3}{2} \cdot 120 \\ -45 + \dfrac{9}{8} \cdot 40 = 0 + 0 \cdot 40 \end{cases}$$

where we used the value process as

$$\begin{cases} x_0 + y_0 \cdot S(u) = x_1^u + y_1^u \cdot S(u) \\ x_0 + y_0 \cdot S(d) = x_1^d + y_1^d \cdot S(d) \end{cases}$$

Solution to problem 11

We prove this by calculate the expectation value

$$\begin{aligned} E^Q\left[(1+r)^{-(k+1)} S_{k+1} \mid \mathcal{F}_k\right] &= (1+r)^{-(k+1)}(p \cdot u + q \cdot d) S_k \\ &= \left(\frac{1}{1+r}\right)^{k+1} \left(\frac{u \cdot (1+r-d)}{u-d} + \frac{d \cdot (u-1-r)}{u-d}\right) S_k \\ &= \left(\frac{1}{1+r}\right)^{k+1} \frac{(1+r)(u-d)}{u-d} S_k \\ &= (1+r)^{-k} S_k \end{aligned}$$

Solution to problem 12

Construct a portfolio (x, y) where x = Cash and y = Stock. Choose x, y replicate the option. x, y constant, so the portfolio is self-financing. The value of the portfolio is then $V = x + yS$.

Idea: If terminal value of portfolio = Option payoff, then present value of portfolio = present value of option, by no-arbitrage. Two assets with identical payoffs must be worth the same at all times; else we would have an arbitrage opportunity!

(a) European Call: Strike = 100. $V_0 = x + 100y$.

$$V_1 = \begin{cases} x + 101y = (101 - 100)^+ = 1 \text{ if } S_1 = 101 \\ x + 99y = (99 - 100)^+ = 0 \text{ if } S_1 = 99 \end{cases}$$

I.e., $x = -99/2$, $y = 1/2$ and the present value of the option is $V_0 = -99/2 + 100/2 = 1/2$. Since $y = 1/2$, the option delta is -1/2, which is the number of shares that needs to be sold to maintain a riskless portfolio.

(b) Cash-or-Nothing Call: Payoff = 100 if $S_1 > 100$, 0 otherwise.

$$V_1 = \begin{cases} x + 101y = 100 \quad \text{if } S_1 = 101 \\ x + 99y = 0 \quad \text{if } S_1 = 99 \end{cases}$$

We have $y = 50$, $x = -99$ x 50 and $V_0 = (100 - 99\}$ x 50 = 50. The option delta is -50. The payoff from the cash-or-nothing call is 100 times the payoff from a European call, i.e. holding a cash-or-nothing call is like holding 100 European calls, at expiry. By no-arbitrage, the value of a cash-or-nothing call must therefore be 100 times the value of a European call at any time, and so V^{CoN} = 100 x V^{EUR} = 50. In general, if the terminal value of asset *A* is guaranteed to be *n* times that of asset *B*, then $V_A = nV_B$ at any time prior to maturity, by no-arbitrage.

Solution to problem 13

We start to calculate the risk neutral probability. The factor u is 1.1 and down 0.5 (as seen in the tree).

$$q = \frac{1 + r - d}{u - d} = \frac{1 + 0.02 - 0.9}{1.1 - 0.9} = 0.6 .$$

For a put option we have the boundary condition at maturity max($K - S_T$, 0). So from the upper to the lower nodes we have the option price as 0, 6 and 24.

a.) The European value in the middle node is then

$$(0.6 \cdot 0 + 0.4 \cdot 6)/(1 + 0.02) = 2.353$$
$$(0.6 \cdot 6 + 0.4 \cdot 24)/(1 + 0.02) = 12.93$$

We use

$$(q \cdot S_u + (1 - q) \cdot S_d)/(1 + r)$$

The fair value of the European put option is therefore

$$(0.6 \cdot 2.353 + 0.4 \cdot 12.94) /(1 + 0.02) = 6.46$$

b.) Next we calculate the value of the American option. Here we need to compare with the intrinsic value to see if we have an early exercise. We now have

$$\max\{(0.6 \cdot 0 + 0.4 \cdot 6) /(1 + 0.02), 105 - 110\} = 2.353$$
$$\max\{(0.6 \cdot 6 + 0.4 \cdot 24) /(1 + 0.02), 105 - 90\} = 15$$

The fair value of the American put option is therefore

$$(0.6 \cdot 2.353 + 0.4 \cdot 15) /(1 + 0.02) = 7.27$$

c.) For the Barrier option we are kicked out in the upper mid node. So the value is here 0. Therefore we get

$$(0.6 \cdot 0 + 0.4 \cdot 12.94) /(1 + 0.02) = 5.07$$

d.) The boundary for the lookback is different. Here we follow the path backwards to find the optimal value. The boundary will therefore be 0, 15 and 24. This is of European type so there are no early exercise, so

$$(0.6 \cdot 0 + 0.4 \cdot 15)/(1 + 0.02) = 5.88$$
$$(0.6 \cdot 15 + 0.4 \cdot 24) /(1 + 0.02) = 18.24$$

The fair value of the American put option is therefore

$$(0.6 \cdot 5.88 + 0.4 \cdot 18.24) /(1 + 0.02) = 10.61$$

Solution to problem 14

We start with

$$V = e^{-r\delta t}\left(qV_u - (1-q)V_d\right) \qquad (1)$$

with the probability can be given as

$$q = \frac{Se^{r\delta t} - S_d}{S_u - S_d}$$

Under this measure, $E[V] = Ve^{r\delta t,}$ This is just another way or writing equation (1). Expanding to $O(\delta t)$:

$$qV_u - (1-q)V_d = q(V_u - V_d) + V_d = V(1 + r\delta t) \qquad (2)$$

In the problem,

$$S_{u,d} = Se^{\pm\sigma\sqrt{\delta t}} = S\left(1 \pm \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t\right) + h.o.t.$$

and so on

$$q = \frac{1 + r\delta t - \left(1 - \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t\right)}{2\sigma\sqrt{\delta t}} = \frac{1}{2} + \frac{1}{2\sigma}\left(r + \frac{1}{2}\sigma^2\right)\sqrt{\delta t}$$

Also

$$V(S + \delta S, t + \delta t) = V(S,t) + V_S\delta S + V_t\delta t + \frac{1}{2}V_{SS}\left(\delta S\right)^2 + h.o.t.$$

and so

$$\begin{aligned} V_{u,d} &= V + SV_S\left(\pm\sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t\right) + V_t\delta t + \frac{1}{2}\sigma^2 S^2 V_{SS}\delta t \\ &= V \pm \sigma SV_S\sqrt{\delta t} + \left\{\frac{1}{2}\sigma^2 SV_S + V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right\}\delta t \end{aligned}$$

From this, we have

$$V_u - V_d = 2\sigma SV_S\sqrt{\delta t}$$

Substituting all these into (2) gives:

$$V(1+r\delta t) = 2\sigma SV_S\sqrt{\delta t}\left[\frac{1}{2}+\frac{1}{2\sigma}\left(r-\frac{1}{2}\sigma^2\right)\sqrt{\delta t}\right]+V$$
$$-\sigma SV_S\sqrt{\delta t}+\left\{\frac{1}{2}\sigma^2 SV_S + V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right\}\delta t$$
$$= V + \left(\sigma SV_S - \sigma SV_S\right)\sqrt{\delta t}$$
$$+\left\{\left(r-\frac{1}{2}\sigma^2\right)SV_S + \frac{1}{2}\sigma^2 SV_S + V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}\right\}\delta t$$

Canceling out terms yields the Black-Scholes equation:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

Solution to problem 15

The market is incomplete. Let Z denote a random variable which equals u with probability p, 1 with probability q, and d with probability $1 - p - q$. If the market was complete it would be possible to solve the following set of equations for every function $\Phi(z)$ (then $h = (x, y)$ would be a replicating portfolio).

$$\begin{cases} x(1+r) + ysu = \Phi(u) \\ x(1+r) + ys = \Phi(1) \\ x(1+r) + ysd = \Phi(d) \end{cases}$$

Let Φ be given by

$$\Phi(z) = \begin{cases} K & \text{if } z = u \text{ or } z = 1 \\ 2K & \text{if } z = d \end{cases}$$

Solving for x and y using the first two equations gives $x = K/(1 + r)$ and $y = 0$, but this does not satisfy the third equation. This means that there are claims which cannot be replicated in this model and therefore the model is not complete.

Solution to problem 16

To derive the Itô's formula in its most simple form, we can start with a Taylor expansion to the lowest orders for a function of two variables: $F(t, X)$

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 + \frac{\partial^2 F}{\partial t \partial X}dtdX + ...$$

where X is described by the stochastic process given by

$$dX = \mu \cdot dt + \sigma \cdot dW$$

Here μ represent a deterministic drift and σ the volatility. W is a Wiener process with the property $(dW)^2 = dt$. Thus, to the lowest order we get

$$\left(dX\right)^2 = \mu^2 \cdot \left(dt\right)^2 + \sigma^2 \cdot \left(dW\right)^2 + 2 \cdot \mu \cdot \sigma \cdot dt \cdot dW \rightarrow \sigma^2 \cdot dt$$

In the lowest order, we ignore $dtdw$ and $dtdt$. To the lowest order of dF, we then have

$$dF = \left(\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial X^2}\right)dt + \sigma\frac{\partial F}{\partial X}dW$$

Solution to problem 17

Let $Z(t) = W^2(t)$ and use Itô formula

$$\begin{aligned} dZ(t) &= \frac{\partial Z}{\partial W}dW + \frac{1}{2}\frac{\partial^2 Z}{\partial W^2}\left(dW\right)^2 \\ &= 2 \cdot W(t) \cdot dW(t) + \frac{1}{2} \cdot 2 \cdot \left(dW(t)\right)^2 = 2 \cdot W(t) \cdot dW(t) + dt \end{aligned}$$

Integration gives

$$W^2(t) = t + 2 \cdot \int_0^t W(s)dW(s) .$$

I.e.

$$\int_0^t W(s)dW(s) = \frac{1}{2}W^2(t) - \frac{t}{2}$$

Solution to problem 18

a) Let $Z = \ln S$ and use Itô's formula to find the differential of Z. This yields

$$dZ = \frac{1}{S}dS + \frac{1}{2}\left(-\frac{1}{S^2}\right)(dS)^2 = \frac{1}{S}(\alpha \cdot Sdt + \sigma \cdot SdW) - \frac{1}{2}\frac{1}{S^2}\sigma^2 S^2 dt$$
$$= \left(\alpha - \frac{1}{2}\sigma^2\right)dt + \sigma \cdot dW$$
$$Z(0) = \ln(s)$$

Integration gives

$$Z_T = Z_t + \int_t^T \left(\alpha - \frac{1}{2}\sigma^2\right)ds + \int_t^T \sigma W_s = Z_t + \left(\alpha - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_t - W_t)$$

Thus

$$S(t) = s \cdot e^{\left\{\left(\alpha - \frac{1}{2}\sigma^2\right)(T-t) + \sigma \cdot (W_T - W_t)\right\}}.$$

b) From the explicit solution above we see that geometric Brownian motion is log-normally distributed with expectation $(\alpha - \sigma^2/2)(T-t)$ and variance $\sigma^2(T-t)$.

Solution to problem 19

Let $Z(t) = \ln L(t)$ and use Itô's formula to find the differential of Z. This yields

$$dZ(t) = \frac{1}{L(t)}dL(t) + \frac{1}{2}\left(-\frac{1}{L^2(t)}\right)(dL(t))^2 = g(t)dW - \frac{1}{2}g(t)^2 dt$$
$$Z(0) = \ln(1) = 0$$

Integrate this and we get

$$L(t) = \exp\left\{\int_0^t g(s)dW(s) - \frac{1}{2}\int_0^t g^2(s)ds\right\}$$

Remark that $L(t)$ is a Radon-Nikodym derivative. We call the function $g(t)$ the Girsanov kernel.

Solution to problem 20

Let $Z(t)$ = ln $L(t)$ and use Itô's formula to find the differential of Z. This yields

$$\begin{cases} dX(t) = \mu dt + \sigma dW(t) \\ X(0) = x \end{cases}$$

Integration gives

$$X_t = X_0 + \int_0^t \mu ds + \int_0^t \sigma W_s = x + \mu t + \sigma W_t$$

Solution to problem 21

Set $Y = e^{-\mu t X}$ and use Itô lemma

$$\begin{aligned} dY = d\left(Xe^{-\mu t}\right) &= \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial X}dX + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}\left(dX\right)^2 \\ &= -\mu Xe^{-\mu t}dt + e^{-\mu t}\left\{\mu Xdt + \sigma dW\right\} + 0 = \sigma e^{-\mu t}dW \end{aligned}$$

By integrating we get

$$Y(t) - Y(0) = \sigma \int_0^t e^{-\mu s} dW(s) \quad \Rightarrow \quad e^{-\mu t} X(t) - x = \sigma \int_0^t e^{-\mu s} dW(s)$$

Finally

$$X(t) = xe^{\mu t} + \sigma \int_0^t e^{-\mu(t-s)} dW(s)$$

Solution to problem 22

i. From the value process we have

$$V_t(h) = h_t^B B_t + h_t^S S_t = 2B_t S_t$$

Applying Ito's formula to this we get

$$dV_t(h) = 2B_t dS_t + 2S_t dB_t = 2\left(h_t^B dB_t + h_t^S dS_t\right).$$

The portfolio defined by $h_t = (S_t, B_t)$ is therefore not self-financing.

ii. Using Ito's formula and the fact that

$$dS_t = rS_t dt + \sigma S_t dV_t$$

under the martingale measure Q (V denotes a Q-Wiener process) we obtain

$$\begin{aligned} dX_t &= -\beta S_t^{-\beta-1} dS_t + \frac{1}{2}(-\beta)(-\beta-1) S_t^{-\beta-2} (dS_t)^2 \\ &= -\beta r S_t^{-\beta} dt - \beta\sigma S_t^{-\beta} dV_t + \frac{1}{2}\left(\beta+\beta^2\right)\sigma^2 S_t^{-\beta} dt . \\ &= rX_t dt - \beta\sigma X_t dV_t \end{aligned}$$

Since the process has a local rate of return of r under the martingale measure Q it represents a tradable asset.

Solution to problem 23

From the value process we have

$$V_t(h) = h_t^B B_t + h_t^S S_t = \frac{B_t S_t^2}{2}$$

Using Ito's formula on this we get

$$dV_t(h) = \frac{\partial V_t}{\partial B_t} dB_t + \frac{\partial V_t}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^{\,2}} (dS_t)^2 = \frac{S_t^2}{2} dB_t + S_t B_t dS_t + \frac{1}{2}\sigma^2 S_t^2 B_t dt$$
$$= -h_t^0 dB_t + h_t^1 dS_t + \frac{\sigma^2}{2r} S_t^2 dB_t = -\left(1 + \frac{\sigma^2}{r}\right) h_t^0 dB_t + h_t^1 dS_t$$

The portfolio is therefore not self-financing.

Solution to problem 24

It is of course possible to use Black-Scholes and integrate and do a lot of work to find the price as

$$S(0) \cdot e^{-r} = S(0) \cdot e^{-0.06}$$

By the standard technique, we can solve the problem with integration as

$$E^Q\left[S(0) \cdot \frac{S(2)}{S(1)}\right] = S(0) \cdot E^Q\left[\frac{S(2)}{S(1)}\right]$$

study $E^Q[X] = E^Q[S(T_2)/S(T_1)]$

$$E^Q[X] = E^Q\left[\frac{S(T_2)}{S(T_1)}\right]$$

where $T_2 > T_1$. Since

$$S(t) = s \cdot \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(W_T - W_t)\right\}$$

we have

$$X = \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T_2 - T_1) + \sigma(W_2 - W_1)\right\}$$

Let $X = S/S(T_1)$ ==>

$$dX = \frac{1}{S(T_1)} dS = \frac{1}{S(T_1)} \left\{ rSdt + \sigma SdW \right\} = rXdt + \sigma XdW$$

Integrate from T_1 to T_2 ($X(T_1) = 1$) and take expectation value ==>

$$X(T_2) - X(T_1) = r \int_{T_1}^{T_2} X(s) ds + \sigma \int_{T_1}^{T_2} X(s) dW(s)$$

$$E^Q \left[X(T_2) \right] - 1 = r \int_{T_1}^{T_2} E^Q \left[X(s) \right] ds + 0$$

Let $m = m(T_2) = E^Q[X(T_2)]$ and take derivative with suspect to T_2 ==>

$$\begin{cases} m = rm \\ m(T_1) = E^Q \left[\dfrac{S(T_1)}{S(T_1)} \right] = 1 \end{cases}$$

So

$$m = E^Q \left[X \right] = e^{r(T_2 - T_1)} = e^r$$

I.e.,

$$\Pi[t] = e^{-r(T-t)} E^Q \left[S(0) \cdot \frac{S(2)}{S(1)} \right] = S(0) \cdot e^{-r(T-t)} E^Q \left[\frac{S(2)}{S(1)} \right]$$
$$= S(0) \cdot e^{-r(T-t)} E^Q \left[X \right]$$

and

$$\Pi[0] = S(0) \cdot e^{-rT_2} E^Q \left[X \right] = S(0) \cdot e^{-2r} e^r = S(0) \cdot e^{-r}$$

It is, as we can see only the interest rate that matters, all other parameters are redundant. We therefore believe that there exists a more simple solution to this problem. Since $S(t) \cdot e^{-rt}$ is a martingale (under the forward measure, se Analytical Finance Vol. II) we then have

$$S(1) = e^{-r} \cdot E^*_{t=1} \left[S(2) \right]$$

where the sub index means conditioned with respect at the information known at time $t = 1$. I.e.:

$$e^r = E^*_{t=1}\left[\frac{S(2)}{S(1)}\right] = E^*\left[\frac{S(2)}{S(1)}\right]$$

where we used "the theorem of repeated expectation". Therefore, we get:

$$\Pi[0] = e^{-2r}E^*\left[S(0)\cdot\frac{S(2)}{S(1)}\right] = S(0)\cdot e^{-r}$$

Remark! The result is independent of the model we are using.

Solution to problem 25

The put-call-parity is obtained from the fact that

$$\max\{K - S_T, 0\} = K - S_T + \max\{S_T - K, 0\}$$

If we denote by $C(t, S(t))$ the price at time t for a European call option with strike price K and exercise time T written on the stock, and by $P(t, S(t))$ the corresponding put option. Then

$$\begin{aligned}
P(t,S(t)) &= e^{-r(T-t)}E^Q\left[\max\{K-S_T,0\} \mid \mathcal{F}_t\right] \\
&= e^{-r(T-t)}E^Q\left[K-S_T+\max\{S_T-K,0\} \mid \mathcal{F}_t\right] \\
&= e^{-r(T-t)}K - e^{rt}E^Q\left[\frac{S_T}{e^{rT}} \mid \mathcal{F}_t\right] + e^{-r(T-t)}E^Q\left[\max\{S_T-K,0\} \mid \mathcal{F}_t\right] \\
&= e^{-r(T-t)}K - e^{rt}\frac{S_t}{e^{rt}} + C(t,S(t))
\end{aligned}$$

In the third step we use that the discounted stock price is a martingale. This gives the put-call-parity

$$P(t,S(t)) = Ke^{-r(T-t)} + C(t,S(t)) - S_t$$

Solution to problem 26

From the put-call parity for European options we have $S + P_E - C_E = Ke^{-rT}$. We know that $C_A = C_E$, and using the argument of additional flexibility of American type options again, we must have $P_A \geq P_E$. Hence we get $S + P_A - C_A \geq Ke^{-rT}$. or, equivalently $C_A - P_A \leq S - Ke^{-rT}$. We thereby obtain an upper bound for the difference of an American call and the corresponding American put option.

To find the lower bound we construct an arbitrage table assuming that the opposite (strict) inequality is true. We set up the following portfolio: write the put, buy the call, sell the stock short, put K into your bank account. We use T^* to denote either the time of early exercise of the put or expiry, whichever comes earlier. The arbitrage table is:

Portfolio	Current cash flow	$S(T^*) \leq K$	$K < S(T^*)$
Write put	P_A	$-(K - S(T^*))$	0
Buy call	$-C_A$	0	$S(T^*) - K$
Sell stock short	S	$-S(T^*)$	$-S(T^*)$
Lend	$-K$	Ke^{rT^*}	Ke^{rT^*}
Total	$-(C_A - P_A) + (S - K) \quad > 0$	> 0	$> 0.$

(of course in the case that T^* means early exercise we needn't look at $K < S(T^*)$ since a rational financial agent wouldn't exercise the put under these circumstances.) Since all future cash-flows are positive we have constructed an arbitrage portfolio, contradicting our assumption. We may even have exercised an American call early, which was suboptimal. Hence the inequality $S - K \leq CA - PA$ must hold, completing the proof.

Solution to problem 27

The self-financing condition in this model is

$$dV_t^h = h_t^0 dB_t + h_t^1 dS_t$$

where V is the value process.

$$V_t^h = h_t^0 B_t + h_t^1 S_t$$

is the value process associated with the portfolio h.

Solution to problem 28

i. The self-financing condition expressed in terms of the relative portfolio is given by

$$dV_t^h = h_t^0 B_t dB_t + h_t^1 S_t dS_t$$
$$= \left\{ u_t^0 = \frac{h_t^0 B_t}{V_t}, \quad u_t^1 = \frac{h_t^1 S_t}{V_t} \right\}$$
$$= u_t^0 V_t \frac{dB_t}{B_t} + u_t^1 V_t \frac{dS_t}{S_t}$$

Inserting the dynamics of *B* and *S* as well as u^0 and u^1 we obtain

$$dV_t^h = u_t^0 V_t \frac{dB_t}{B_t} + u_t^1 V_t \frac{dS_t}{S_t} = V_t \frac{rB_t dt}{2B_t} + V_t \frac{rS_t dt + \sigma S_t dW_t}{2S_t}$$
$$= \frac{r+\alpha}{2} V_t dt + \frac{\sigma}{2} V_t dW_t$$

ii. Taking $Z = \ln(V)$ and Ito, we found that the value process *V* thus follows GBM with the solution

$$dZ = \frac{\partial Z}{\partial t} dt + \frac{\partial Z}{\partial V} dV + \frac{1}{2} \frac{\partial^2 Z}{\partial V^2} (dV)^2 =$$
$$= \frac{1}{V} \left(\frac{r+\alpha}{2} \cdot V dt + \frac{\sigma}{2} \cdot V dW \right) - \frac{1}{2} \frac{1}{4} \frac{1}{V^2} \sigma^2 \cdot V^2 dt$$
$$= \left(\frac{r+\alpha}{2} - \frac{1}{2} \frac{1}{4} \sigma^2 \right) dt + \frac{\sigma}{2} dW$$

Integration from 0 to *t* and taking exponent gives

$$V_t = V_0 \exp\left\{ \left(\frac{r+\alpha}{2} - \frac{1}{2} \frac{\sigma^2}{4} \right) t + \frac{\sigma}{2} W_t \right\}$$

Solution to problem 29

Introduce a stochastic process

$$\begin{cases} dX(s) = \sigma \cdot dW(s) \\ X(t) = x \end{cases}$$

And applying the Itô formula on the function $F(t, X)$

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) dt + \sigma \frac{\partial F}{\partial x} dW = \sigma \frac{\partial F}{\partial x} dW$$

Now, if we integrate and take the expectation value, we end up with the Feynman-Kač representation

$$F(t,x) = E_{t,x}^{Q}\left[X_T^2 \right]$$

Now, let $Z = X^2$ and apply the Itô formula on Z

$$dZ = \frac{\partial Z}{\partial X} dX + \frac{1}{2}\frac{\partial^2 Z}{\partial X^2}(dX)^2 = 2 \cdot X_t dX + \frac{1}{2} \cdot 2 \cdot (dX)^2 = \sigma^2 dt + 2 \cdot \sigma \cdot XdW$$

Integration gives

$$Z(T) - Z(t) = \int_t^T \sigma^2 ds + \int_t^T 2 \cdot \sigma \cdot XdW = \sigma^2 (T-t) + 2 \cdot \sigma \int_t^T XdW$$

Finally, we take the expectation value and get

$$E\left[Z(T)\right] = E\left[Z(t)\right] + \sigma^2(T-t) + 2\sigma \int_t^T E\left[X\right] dW = x^2 + \sigma^2 (T-t)$$

We therefore have the following solution

$$F(t,x) = x^2 + \sigma^2 (T-t)$$

We can also find the solution by using $dX = \sigma \cdot dW$ and $X_T = x + \sigma\left[W_T - W_t\right]$

$$F(t,x) = E_{t,x}^{Q}\left[X_T^2\right] = Var\left[X_T\right] + \left\{E_{t,x}^{Q}\left[X_T\right]\right\}^2 = \sigma^2 (T-t) + x^2$$

Solution to problem 30

Suppose $F(t, X)$ solves the PDE, where $dX = \sigma X dW$ and $X(0) = x$. Using Itô we get

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 = \left(F_t + \frac{1}{2}x^2\sigma^2 F_{XX}\right)dt + \sigma XF_X\, dW$$
$$= \sigma XF_X\, dW$$

Integration gives

$$X^2 = F(T, X(T)) = F(t, X(t)) + \sigma\int_t^T x\frac{\partial F}{\partial X}dW(s)$$

If we now take the expectation value we get the Feynman-Kač formula

$$F(t,x) = E^Q_{t,x}\left[X_T^2\right]$$

As always, to calculate such an expectation, we need the dynamics of $Z = X^2$ and use the Itô Lemma. We then get

$$\begin{cases} dZ = 2\cdot X_t dX + \frac{1}{2}\cdot 2\cdot(dX)^2 = \sigma^2 X^2 dt + 2\sigma X^2 dW = \sigma^2 Z dt + 2\sigma Z dW \\ Z(0) = X^2(0) = x^2 \end{cases}$$

We now integrate

$$Z(T) = Z(t) + \sigma^2\int_t^T Zds + 2\sigma\int_t^T ZdW$$

This is an integral equation and the easiest way to solve this is to convert it to a differential equation. The standard technique to solve this equation is to is, first to define $m = E[Z]$ and then take the derivative with respect to time. We then get the following ordinary differential equation

$$\begin{cases} \dfrac{dm}{dt} = -\sigma^2 m \\ m(T) = x^2 \end{cases}$$

This gives the solution to the partial differential equation

$$F(t,x) = m = x^2 e^{\sigma^2 (T-t)}$$

Solution to problem 31

Suppose $F(t, X)$ solves the PDE where $dX = \sigma X dW$ and $X(0) = x$. Using Itô we get

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 = \left(F_t + \frac{1}{2}x^2\sigma^2 F_{XX}\right)dt + \sigma X F_X dW \\ &= rFdt + \sigma X F_X dW \end{aligned}$$

The stochastic process

$$\begin{cases} dX_t = (aX_t + b_t)dt + \sigma_t dW_t \\ X_0 = x_0 \end{cases}$$

have the solution

$$X_t = e^{at}x_0 + \int_0^t e^{a(t-s)} b_s ds + \int_0^t e^{a(t-s)} \sigma_s dW_s$$

Integrate ($b = 0$, $a = r$, $x_0 = x^2$, $\sigma_t = \sigma X F_X$ $X_t = F$)

$$F(t, X(t)) = e^{-r(T-t)} X^2(T) + \sigma \int_t^T e^{-r(T-s)} X \frac{\partial F}{\partial X} dW(s)$$

and take the expectation value

$$F(t,x) = e^{-r(T-t)} E_{t,x}^{Q}\left[X_T^2\right]$$

As always, we need the dynamics of $Z = X^2$ and by using Itô we get

$$\begin{cases} dZ = 2 \cdot X_t dX + \frac{1}{2} \cdot 2 \cdot (dX)^2 = \sigma^2 X^2 dt + 2\sigma X^2 dW = \sigma^2 Z dt + 2\sigma Z dW \\ Z(T) = X^2(T) = x^2 \end{cases}$$

Integrate

$$Z(t) = Z(T) - \sigma^2 \int_t^T Z ds - 2\sigma \int_t^T Z dW$$

and taking the expectation value

$$E[Z] = x^2 - \sigma^2 \int_t^T E[Z] ds$$

Take the derivative and let $E[Z] = m$ we get the following ordinary differential equation:

$$\begin{cases} \frac{dm}{dt} = -\sigma^2 m \\ m(T) = x^2 \end{cases}$$

This gives the solution to the partial differential equation:

$$F(t, x) = e^{-r(T-t)} m = e^{-r(T-t)} x^2 e^{\sigma^2 (T-t)} = x^2 e^{(\sigma^2 - r)(T-t)}$$

Solution to problem 32

This is exactly as Black-Scholes, but easier because the normal distribution:

$$\Pi_C = e^{-r(T-t)}E^Q\left[\Phi(T)\right] = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty}\Phi(T)\cdot e^{-\frac{(f_T-f_t)^2}{2\sigma^2(T-t)}}df_T$$

$$= \left\{ f_T = f_t + \sigma\sqrt{T-t}\cdot z \right\} = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty}\Phi(T)\cdot e^{-\frac{\left(\sigma\sqrt{T-t}\cdot z\right)^2}{2\sigma^2(T-t)}}\sigma\sqrt{T-t}dz$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(T)\cdot e^{-\frac{z^2}{2}}dz$$

Here

$$\Phi(T) = \begin{cases} (f_T - K)^+ & \text{for a Call} \\ (K - f_T)^+ & \text{for a Put} \end{cases}$$

For the Call then we have

$$\Pi_C(t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f_T - K)^+\cdot e^{-\frac{z^2}{2}}dz = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f_t + \sigma\sqrt{T-t}\cdot z - K)^+\cdot e^{-\frac{z^2}{2}}dz$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{z_0}^{\infty}(f_t + \sigma\sqrt{T-t}\cdot z - K)\cdot e^{-\frac{z^2}{2}}dz = A - B$$

Set $f_t = F$ and with $z_0 = \frac{(F-K)}{\sigma\sqrt{T-t}}$ we get

$$A = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}(F-K)\int_{z_0}^{\infty}e^{-\frac{z^2}{2}}dz = e^{-r(T-t)}(F-K)\cdot N[z_0]$$

$$B = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{z_0}^{\infty}\sigma\sqrt{T-t}\cdot z\cdot e^{-\frac{z^2}{2}}dz = e^{-r(T-t)}\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{z_0^2}{2}}$$

Then, the fair values of call C is given by

$$C = e^{-r(T-t)}\left[\left(F-K\right)\cdot N(d_1) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-d_1^2/2}\right]$$

Solution to problem 33

(a) If we note that: $\max\{S_T, K\} = K + \max\{S_T - K, 0\}$ we can write down the price of the option as:

$$\Pi_t = e^{-r(T-t)} E^Q \left[K + \max\{S_T - K\} \mid F_t \right] = e^{-r(T-t)} K + C(t,S,K,T,r,\sigma)$$

where $C(t,S,K,T,r,\sigma)$ denotes the Black-Scholes price of a European call option with strike K at maturity T. Therefore

$$\Pi_t = e^{-r(T-t)} K + S \cdot N\left[d_1(t,S)\right] - e^{-r(T-t)} K \cdot N\left[d_2(t,S)\right]$$

Finally, with $N(-x) = 1 - N(x)$ we get.

$$\Pi_t = e^{-r(T-t)} K \cdot N\left[-d_2(t,S)\right] + S \cdot N\left[d_1(t,S)\right]$$

(b) The price of the option at time $t \in [0, T_0]$ is given by

$$\begin{aligned}
\Pi_t &= e^{-r(T-t)} E^Q \left[\max\left\{S_T, S_{T_0}\right\} \mid F_t \right] \\
&= e^{-r(T-t)} E^Q \left[E^Q \left[\max\left\{S_T, S_{T_0}\right\} \mid F_{T_0} \right] \mid F_t \right] \\
&= \left\{ S_T = S_{T_0} \exp\left[\left(r - \frac{1}{2}\sigma^2 \right)(T - T_0) + \sigma\left(V_T - V_{T_0}\right) \right] \right\} \\
&= e^{-r(T-t)} E^Q \left[E^Q \left[S_{T_0} \max\left\{ \exp\left[\left(r - \frac{1}{2}\sigma^2 \right)(T - T_0) + \sigma\left(V_T - V_{T_0}\right) \right], 1 \right\} \mid F_{T_0} \right] \mid F_t \right] \\
&= e^{-r(T-t)} E^Q \left[\frac{S_{T_0}}{e^{-r(T-T_0)}} \times \right. \\
&\quad \left. \times e^{-r(T-T_0)} E^Q \left[\max\left\{ \exp\left[\left(r - \frac{1}{2}\sigma^2 \right)(T - T_0) + \sigma\left(V_T - V_{T_0}\right) \right], 1 \right\} \mid F_{T_0} \right] \mid F_t \right]
\end{aligned}$$

$$\begin{aligned}
&= e^{rt} E^Q \left[\frac{S_{T_0}}{e^{rT_0}} \Pi^a \left(T_0, 1, 1, T, r, \sigma\right) \mid F_t \right] = e^{rt} E^Q \left[\frac{S_{T_0}}{e^{rT_0}} \mid F_t \right] \Pi^a \left(T_0, 1, 1, T, r, \sigma\right) \\
&= S_t \Pi^a \left(T_0, 1, 1, T, r, \sigma\right) = S_t \left(e^{-r(T-T_0)} N\left[-d_2(T_0, 1)\right] + N\left[d_1(T_0, 1)\right] \right)
\end{aligned}$$

For the second last equality we have used the fact that S_t/B_t is Q-martingale. Π^a denotes the price at time t of the option in exercise (a).

Solution to problem 34

The price of the claim at time $t \in [0, T]$ is given by

$$\Pi(t) = e^{-r(T-t)} E^Q\left[\sqrt{S_T} \mid \mathcal{F}_t\right]$$

The dynamics of S under Q are given by

$$dS_t = rS_t dt + \sigma S_t dW_t; \; S(0) = s_0$$

where V is a Q-Wiener process. Let $Z = \sqrt{S}$ and use Itô's formula to find the differential of Z

$$dZ = \frac{1}{2}\frac{1}{\sqrt{S}}dS + \frac{1}{2}\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{S^{3/2}}(dS)^2 = \left(\frac{1}{2}r - \frac{1}{8}\sigma^2\right)Zdt + \frac{1}{2}\sigma ZdW$$

Integrating this we obtain

$$Z_T = Z_t + \int_t^T \left(\frac{1}{2}r - \frac{1}{8}\sigma^2\right)Z_s ds + \frac{1}{2}\int_t^T \sigma_s Z_s dW_s$$

Now take the conditional expectation

$$E\left[Z_T \mid \mathcal{F}_t\right] = Z_t + \int_t^T \left(\frac{1}{2}r - \frac{1}{8}\sigma^2\right)E\left[Z_s \mid \mathcal{F}_t\right]ds + 0$$

Let $m = E\left[Z_T \mid \mathcal{F}_t\right]$ and take derivatives with respect to T

$$\begin{cases} m = \left(\frac{1}{2}r - \frac{1}{8}\sigma^2\right)m \\ m(t) = Z_t \end{cases}$$

Solving the ODE above we get

$$m_T = Z_t e^{\left(\frac{1}{2}r - \frac{1}{8}\sigma^2\right)(T-t)}$$

The price of the claim is given by $e^{-r(T-t)}m(T)$, i.e.

$$\Pi_t[X] = \sqrt{S_t} \cdot e^{-\left(\frac{r}{2}+\frac{\sigma^2}{8}\right)(T-t)}$$

Solution to problem 35

We know that $\Pi(t) = V(t; h) = F(t, S_t)$. Therefore using Itô's formula on F we obtain

$$dV = dF = F_t dt + F_s dS + \frac{1}{2}F_{ss}\left(dS\right)^2 = \left(F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}\right)dt + F_s dS$$

Now comparing with the equation for a self-financing process

$$dV(h,t) = h^0(t)dB(t) + h^1(t)dS(t)$$

we see that we must have

$$h^0 rB = F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}$$

and

$$h^1 = F_s = \Delta$$

Using Black-Scholes equation and that $B\{t) = e^{rt}$ the expression for h^0 can be written as

$$h^0 = \frac{F - SF_s}{e^{rt}} = \frac{F - S \cdot h^1(t)}{e^{rt}} = \frac{F - S \cdot \Delta}{e^{rt}}$$

Solution to problem 36 of delta

Using Black-Scholes formula for a call option is given by:

$$C = S \cdot N(d_1) - X \cdot e^{-rT} \cdot N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2) \cdot T}{\sigma \cdot \sqrt{T}}, \qquad d_2 = d_1 - \sigma \cdot \sqrt{T}$$

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S \cdot \sigma \cdot \sqrt{T}}$$

I.e.

$$\begin{aligned}
\Delta = \frac{\partial C}{\partial S} &= \frac{\partial}{\partial S}\left[S \cdot N(d_1)\right] - X \cdot e^{-rT} \cdot \frac{\partial}{\partial S} N(d_2) \\
&= N(d_1) + S \cdot N'(d_1) \cdot \frac{\partial d_1}{\partial S} - X \cdot e^{-rT} \cdot N'(d_2) \cdot \frac{\partial d_2}{\partial S} \\
&= N(d_1) + S \cdot N'(d_1) \cdot \frac{1}{S \cdot \sigma \cdot \sqrt{T}} - X \cdot e^{-rT} \cdot N'(d_2) \cdot \frac{1}{S \cdot \sigma \cdot \sqrt{T}} \\
&= N(d_1) + \frac{1}{S \cdot \sigma \cdot \sqrt{T}}\left[S \cdot N'(d_1) - X \cdot e^{-rT} \cdot N'(d_2)\right]
\end{aligned}$$

But

$$\begin{aligned}
X \cdot e^{-rT} \cdot N'(d_2) = X \cdot e^{-rT} \cdot N'(d_1 - \sigma \cdot \sqrt{T}) \\
&= X \cdot e^{-rT} \cdot N'(d_1) \cdot e^{d_1 \cdot \sigma \cdot \sqrt{T}} \cdot e^{-\sigma^2 \cdot T/2} \\
&= X \cdot e^{-rT} \cdot N'(d_1) \cdot e^{-\sigma^2 \cdot T/2} \cdot \frac{S}{X} \cdot e^{rT} \cdot e^{\sigma^2 \cdot T/2} \\
&= S \cdot N'(d_1)
\end{aligned}$$

Finally we get

$$\Delta = N(d_1)$$

Solution to problem 37

From problem 35 we get (With the boundary condition to B&S given by $F(T, s) = s^2$):

$$\begin{cases} h^0(t) = -S_t^2 e^{(r+\sigma^2)(T-t)-rt} \\ h^1(t) = 2 \cdot S_t \, e^{(r+\sigma^2)(T-t)} \end{cases}$$

Solution to problem 38

We have three sources of randomness (W, N^1 and N^2) and only two risky assets. The meta theorem or rule of thumb thus tells us that the model should be free of arbitrage ("more randomness than papers"), but not complete (to few papers compared to the number of sources of randomness). A modification of the model which will make it both free of arbitrage and complete is to remove the Poisson process, say N^2, and let both stocks be driven by W and N^1.

Solution to problem 39

The price of the claim is given by

$$\Pi_t = e^{-r(T-t)} E^Q \left[\sqrt{S_T} I_{\{S_T > K\}} \mid \mathcal{F}_t \right]$$

Since $S_T = S_t e^y$ where y is $N[(r - \sigma^2/2)(T - t), \sigma^2(T - t)]$ this can be written as

$$\Pi_t = e^{-r(T-t)} \int_{\ln\{K/S_t\}}^{\infty} \sqrt{S_T} e^{y/2} \varphi(z) dz$$

where φ denotes the normal probability density distribution and where we have used $S_T > K$ ==> $s \cdot e^y > K$ ==> $y > \ln\{K/S\}$. We also have

$$y = r \cdot \tau + \sigma \cdot \sqrt{\tau} z \quad \Rightarrow \quad z_0 = \frac{\ln\{K/S\} - r \cdot \tau}{\sigma \cdot \sqrt{\tau}}$$

Now, it's time to integrate using

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

I.e.,

$$\Pi = e^{-r\cdot\tau}\int_{z_0}^{\infty}\sqrt{s}\cdot e^{\frac{1}{2}\left(r\tau+\sigma\sqrt{\tau}z\right)}\varphi(z)dz =$$

$$= \frac{\sqrt{s}\cdot e^{-(r-r/2)\tau}}{\sqrt{2\pi}}\int_{z_0}^{\infty}e^{\sigma\sqrt{\tau}z/2-z^2/2}dz = \frac{\sqrt{s}}{\sqrt{2\pi}}e^{-\frac{1}{8}\left(4r+\sigma^2\right)(T-t)}\int_{z_0}^{\infty}e^{-\frac{1}{2}\left(z-\frac{1}{2}\sigma\sqrt{\tau}\right)^2}dz =$$

$$= \sqrt{s}\cdot e^{-\frac{1}{8}\left(4r+\sigma^2\right)(T-t)}\cdot N\left[-z_0+\frac{1}{2}\sigma\sqrt{\tau}\right]$$

So

$$\Pi_t = \sqrt{S_t}\cdot e^{-\frac{1}{8}\left(4r+\sigma^2\right)(T-t)}N\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{S_t}{K}\right\}+r\left(T-t\right)\right\}\right)$$

Solution to problem 40

The price of the claim is given by

$$\Pi_t = e^{-r(T-t)}E^Q\left[S_T^2 I_{\{S_T>K\}}\,|\mathcal{F}_t\right]$$

Since $S_T = S_t e^y$ where $y \in N[(r - \sigma^2/2)(T - t), \sigma^2(T - t)]$:

$$Z(t) = \ln s + \left(r - \frac{1}{2}\sigma^2\right)\left(T-t\right) + \sigma\left(W_T - W_t\right)$$

I.e.

$$S(t) = s\cdot\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)\left(T-t\right)+\sigma\left(W_T-W_t\right)\right\} = se^y$$

If we introduce

$$r = r - \frac{1}{2}\sigma^2, \qquad \tau = T - t \qquad v = W_T - W_t = \sqrt{\tau}z$$

this can be written as

$$\Pi_t = e^{-r(T-t)}\int_{z_0}^{\infty}S_t^2 e^{2y}\varphi(y)dy$$

where φ denotes the probability density function of a $N[(r - \sigma^2/2)(T - t), \sigma^2(T - t)]$-

distribution and where we have used $S_T > K$ ==> $s{\cdot}e^y > K$ ==> $y > \ln\{K/S\}$. We also have

$$y = r\cdot\tau + \sigma\cdot\sqrt{\tau}z \quad \Rightarrow \quad z_0 = \frac{\ln\{K/S\} - r\cdot\tau}{\sigma\cdot\sqrt{\tau}}$$

Now, it's time to integrate and use

$$\varphi(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$$

I.e.,

$$\Pi = e^{-r\cdot\tau}\int_{z_0}^{\infty} s^2\cdot e^{2\cdot r\cdot\tau+2\cdot\sigma\cdot\sqrt{\tau}z}\varphi(z)dz =$$

$$= \frac{s^2\cdot e^{-r\cdot\tau}}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{\left(2r-\sigma^2\right)\tau+2\sigma\sqrt{\tau}z-z^2/2}dz = \frac{s^2\cdot e^{-r\cdot\tau}}{\sqrt{2\pi}}e^{2r\tau-\sigma^2\tau}\int_{z_0}^{\infty} e^{2\sigma\sqrt{\tau}z-z^2/2}dz =$$

$$= \frac{s^2\cdot e^{\left(r+\sigma^2\right)\tau}}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}\left(z-2\sigma\sqrt{\tau}\right)^2}dz = s^2\cdot e^{\left(r+\sigma^2\right)(T-t)}\cdot N\left[-z_0+2\sigma\sqrt{\tau}\right]$$

So

$$\Pi_t = e^{\left(r+\sigma^2\right)(T-t)}S_t^2 N\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{S_t}{K}\right\}+\left(r+\frac{3}{2}\sigma^2\right)(T-t)\right\}\right)$$

Solution to problem 41

(a) See Lecture Notes

(b) The value process $V(t)$ is given by

$$V(t) = h^B(t)B(\mathrm{t}) + h^S(t)S(t)$$

To be a self-financing portfolio strategy, the value process V must satisfy

$$dV(t) = h^B(t)dB(\mathrm{t}) + h^S(t)dS(t)$$

In order for the relative portfolios, the value process V must satisfy

$$dV = V\left[u^B \frac{dB}{B} + u^S \frac{dS}{S}\right] = V\left[u^B \frac{rBdt}{B} + u^S \frac{1}{S}\left(\alpha Sdt + \sigma SdW\right)\right]$$
$$= V\left[\frac{1}{2} rdt + \frac{1}{2}\alpha dt + \frac{1}{2}\sigma dW\right] = \frac{1}{2}\left(r+\alpha\right)Vdt + \frac{1}{2}\sigma VdW$$

where

$$u_t^B = \frac{h^B B_t}{V_t}, \qquad u_t^S = \frac{h^S S_t}{V_t}, \qquad u_t^B + u_t^S = 1$$

To solve the above SDE we use $V_0 = v_0$ and define $Z = \ln(V)$ and use Itô:

$$dZ = \frac{1}{V}dV - \frac{1}{2}\frac{1}{V^2}\left(dV\right)^2 = \frac{1}{2}\left(r+\alpha\right)dt + \frac{1}{2}\sigma dW - \frac{1}{8}\sigma^2 dt$$
$$= \frac{1}{2}\left(r + \alpha - \frac{1}{4}\sigma^2\right)dt + \frac{1}{2}\sigma dW$$

Integration gives

$$Z = Z_0 + \frac{1}{2}\left(r + \alpha - \frac{1}{4}\sigma^2\right)t + \frac{1}{2}\sigma W$$

and

$$V_t = v_0 \exp\left\{\frac{1}{2}\left(r + \alpha - \frac{1}{4}\sigma^2\right)t + \frac{1}{2}\sigma W\right\}$$

We also know that

$$S_t = s_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W\right\}$$

The portfolio is now easily found to be ($B = e^{rt}$):

$$h_t^B = \frac{u^B V_t}{B_t} = \frac{v_0}{2}\exp\left\{\left[\frac{1}{2}\left(\alpha + r\right) - \frac{1}{8}\sigma^2 - r\right]t + \frac{1}{2}\sigma W_t\right\} =$$
$$= \frac{v_0}{2}\exp\left\{\left[\frac{1}{2}\left(\alpha - r\right) - \frac{1}{8}\sigma^2\right]t + \frac{1}{2}\sigma W_t\right\}$$

and

$$h_t^S = \frac{u^S V_t}{S_t} = \frac{1}{2}\frac{v_0}{s_0}\exp\left\{\left[\frac{1}{2}(r+\alpha)-\frac{1}{8}\sigma^2\right]t+\frac{1}{2}\sigma W_t-\left(r-\frac{1}{2}\sigma^2\right)t-\sigma W\right\}$$
$$= \frac{1}{2}\frac{v_0}{s_0}\exp\left\{\left[\frac{1}{2}(\alpha-r)+\frac{3}{8}\sigma^2\right]t-\frac{1}{2}\sigma W_t\right\}$$

Solution to problem 42

a) Our CEO wants to construct a portfolio whose value at time T will equal $-A \cdot \ln(S_T/K)$, as this portfolio will exactly neutralize her contract. She will have eliminated the uncertainty present in her contract, and will have no net cash flow at time T. The market is complete (assuming she can trade in the company stock), so the cost of implementing the hedge must be

$$e^{-rT}E\left[-A\ln\left(S_T/K\right)\right]$$

b) Since

$$S_T = S_0 e^{\left(r-\sigma^2/2\right)T+\sigma B_T}$$

we have that

$$e^{-rT}E\left[-A\ln\left(S_T/K\right)\right] = -Ae^{-rT}E\left[\ln\left(S_0/K\right)+\left(r-\frac{1}{2}\sigma^2\right)T+\sigma B_T\right]$$
$$= -Ae^{-rT}\left[\ln\left(S_0/K\right)+\left(r-\frac{1}{2}\sigma^2\right)T\right]$$

c) Substituting the parameters, the above expression equals

$$-100000\cdot e^{-2\cdot 0.03}\left[\ln(12/10)+2\cdot(0.03-0.3\cdot 0.3/2)\right] = -14345$$

Note that the hedging cost is negative, which means that the net result of hedging is that she receives an immediate payment of \$14345. She has basically traded a certain immediate payment for an uncertain (though possibly larger) future payment/debt.

Solution to problem 43

The price of a binary asset-or-nothing call is given by

$$\Pi_t = e^{-r(T-t)} E^Q\left[S_T I_{\{S_T > K\}} \mid \mathcal{F}_t \right] = e^{-r(T-t)} \int_{Z_0}^{\infty} s e^z \varphi(z) dz$$

where φ denotes the density of a $N[(r - \sigma^2/2)(T - t), \sigma^2(T - t)]$-distribution. Now use that the density function for a $N[(r - \sigma^2/2)(T - t), \sigma^2(T - t)]$-distributed and complete the square in the exponent. This yields

$$\Pi_t = s \int_{Z_0}^{\infty} \psi(z) dz$$

where ψ denotes the density of a $N[(r + \sigma^2/2)(T - t), \sigma^2(T - t)]$-distribution. We thus have that

$$\begin{aligned}\Pi_t\left[BAC_T\right] &= sQ\left(Z > Z_0\right) \\ &= s\left[1 - N\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{K}{s}\right\} - \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}\right)\right]. \\ &= sN\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{s}{K}\right\} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}\right)\end{aligned}$$

Where we have used one of the hints to obtain the last equality. We recognize the first half of Black-Scholes formula for the price of a European call option. For the binary cash-or-nothing call we have

$$\begin{aligned}\Pi_t\left[BCC_T\right] &= e^{-r(T-t)} E^Q\left[K I_{\{S_T > K\}} \mid \mathcal{F}_t \right] = e^{-r(T-t)} K Q_{t,s}\left(S_T > K\right) \\ &= e^{-r(T-t)} K\left(1 - Q_{t,s}\left(se^Z \le K\right)\right)\end{aligned}$$

Where $Z \in N[(r - \sigma^2/2)(T - t), \sigma^2(T - t)]$. Rewriting a bit gives

$$\Pi_t\left[BCC_T\right]=e^{-r(T-t)}K\left[1-N\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{K}{s}\right\}-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right\}\right)\right]$$
$$=e^{-r(T-t)}KN\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{s}{K}\right\}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right\}\right)$$

where we have used one of the hints to obtain the last equality. We recognize the second half of Black-Scholes formula for the price of a European call option.

Solution to problem 44

The Black-Scholes PDE is given by:

$$\frac{\partial F(t,S)}{\partial t}+rS\frac{\partial F(t,S)}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2F(t,S)}{\partial S^2}=rF(t,S)$$

Or

$$\Theta+rS\Delta+\frac{1}{2}\sigma^2S^2\Gamma=rF(t,S)$$

i.e.

$$F(t,S)=\frac{\Theta+rS\Delta+\frac{1}{2}\sigma^2S^2\Gamma}{r}$$
$$=\frac{-13.76591+0.06\cdot100\cdot0.597866+\frac{1}{2}0.40^2100^20.013659}{0.06}$$
$$=\frac{-13.76591+3.587196+10.9272}{0.06}$$
$$=12.4748$$

Solution to problem 45

Her we have to remember the formula for the time-dependent volatility shown in the Black-Scholes section, i.e.

$$\sigma_c^2=\frac{1}{T-t}\int_t^T\sigma^2(\tau)d\tau$$

giving

$$\sigma_c^2 = \int_0^1 \sigma^2(t)dt = 0.25^2 \int_0^1 \frac{1}{(3+t)^2} dt = 0.25^2 \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{192}$$

So the “constant” or average volatility is

$$\sigma = \sqrt{\frac{1}{192}} = \frac{\sqrt{3}}{24}$$

So the standard deviation of the logarithm of the exchange rate is $\sqrt{3}/24$ after a year. The forward price is: $0.98 \cdot e^{0.03 - 0.05} = 0.98 \cdot e^{-0.02}$ so the option price is

$$980 \cdot e^{-0.05} N(d_1) - 1100 \cdot e^{-0.03} N(d_2) \approx 0.842653 \text{ EUR}$$

where

$$d_1 = \frac{\ln(980/1100) + 0.03 - 0.05 + \frac{1}{2 \cdot 192}}{\frac{\sqrt{3}}{24}} = -1.84164$$

$$d_2 = d_1 - \frac{\sqrt{3}}{24} = -1.91381$$

$$\Rightarrow$$

$$N(d_1) = 0.032764$$

$$N(d_2) = 0.027822$$

Solution to problem 46

The result depends upon how the binomial tree is constructed. Especially since the time period is long. If we use a 3 step binomial tree we get. First we need the risk-neutral probabilities. Now, with two rates we use

$$\begin{cases} u = e^{(r_d - r_f)\Delta t + \sigma\sqrt{\Delta t}} = e^{(0.03-0.05)/3 + 0.06\sqrt{1/3}} = 1.02837 \\ d = e^{(r_d - r_f)\Delta t - \sigma\sqrt{\Delta t}} = e^{(0.03-0.05)/3 - 0.06\sqrt{1/3}} = 0.95953 \end{cases}$$

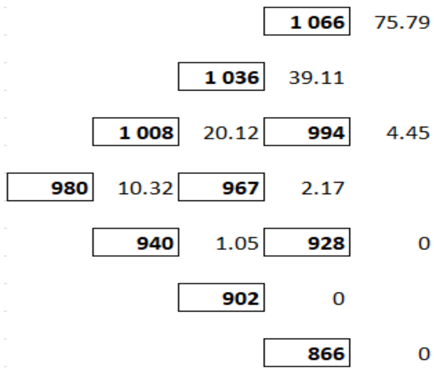
Then

$$q = \frac{e^{(r_d - r_f)\Delta t} - d}{u - d} = 0.49134$$

We then get the tree below, where e.g.

$$39.11 = (75.79 \cdot 0.49134 + 4.45 \cdot (1 - 0.49133)) \cdot e^{(-0.03/3)}$$

and so on…



So, the option price is 10.32 EUR.

Solution to problem 47

The value of the total portfolio is given in the last row of the table below. From this we see that you will have gained 5.47 from selling the option and then delta hedging it.

Time	0	1/3	2/3	1
Stock price	100	114	104	108
Option price	6.78	12.87	3.81	0
Δ of option	0.4861	0.7313	0.4504	-
Value of option position	-6.78	-12.87	-3.81	0
Value of stock position	48.61	83.37	46.84	48.64
dΔ	0.4861	0.2452	-0.2809	-
Bank	-41.83	-70.49	-42.46	-43.17
Value of total portfolio	0	0.01	0.57	5.47

Below you will find explanations of how the values have been computed. The notation $\Delta t = 1/3$ is used. $e^{r\Delta t} = 1.0168$.

<u>At $t = 0$</u>

1. Value of option position = (-1)·option price = -6.78 (short position)
2. Value of stock position = Δ·stock price = 0.4861 · 100 = 48.61
3. $d\Delta(t) = \Delta(t)$ (i.e. the change in the delta of the option) = 0.4861
4. Bank(t) = 6.78 – 48.61 = -41.83
5. Total value = 0

<u>At $t = 1/3$</u>

1. Value of option position = (-1).option price = -12.87
2. Value of stock position = Δ.stock price = 0.7313 · 114 = 83.37
3. $d\Delta(t) = \Delta(t) - \Delta(t - \Delta t)$ = 0.7313 - 0.4861 = 0.2452
4. Bank(t) = Bank(t - Δt)·$e^{r\Delta t}$ - dΔ·stock price = -42.53 – 0.2452·114 = -70.49
5. Value of total portfolio = Value of option position + Value of stock position + Bank = -12.87 + 83.37 – 70.49 = 0.01

<u>At $t = 2/3$</u>

1. Value of option position = (-1).option price = -3.81
2. Value of stock position = Δ.stock price = 0.4504 · 104 = 46.84
3. $d\Delta(t) = \Delta(t) - \Delta(t - \Delta t)$ = 0.4504 - 0.7313 = -0.2809
4. Bank(t) = Bank(t - Δt)·$e^{r\Delta t}$ - dΔ·stock price = -71.67 + 0.2809.104 = -42.46
5. Value of total portfolio = Value of option position + Value of stock position + Bank = -3.81 + 46.84 – 42.46 = 0.57

<u>At $t = 1$</u>

1. Value of option position = (-1).option price = 0
2. Value of stock position = Δ.stock price = 0.4504 · 104 = 48.64
3. Bank(t) = Bank(t - Δt)·$e^{r\Delta t}$ = -43.17
4. Value of total portfolio = Value of option position + Value of stock position + Bank = 0 + 48.64 – 43.17 = 5.47

Solution to problem 48

If we note that $|S_T - K| = 2 \cdot \max\{ S_T - K, 0\} - S_T + K$ we can write down the price of the option as

$$\Pi(t) = e^{-r(T-t)} E^Q \left[2 \cdot \max\{S_T - K, 0\} - S_T + K \,|\, \mathcal{F}_t \right]$$
$$= 2 \cdot C(t, S_t, K, T, r, \sigma) - S_t + Ke^{-r(T-t)}$$

Here *C*(*t, s, K, T, r, σ*) denotes the standard Black-Scholes price at time *t* of a European call option with exercise price *K* and expiry date *T*, when the current price of the underlying is *s*, the interest rate is *r*, and the volatility of the underlying is *σ*. Using one of the formulas on the last page of we obtain

$$\Pi(t) = 2\left(S_t N(d_1) - Ke^{-r(T-t)} N(d_2) \right) - S_t + Ke^{-r(T-t)}$$

Solution to problem 49

Denote the payoff function by Φ and note that

$$\Phi(S_T) = \max\left\{x_1 - S_T, 0\right\} + \max\left\{S_T - x_2, 0\right\}$$

The price of the contract is therefore

$$\Pi_t = e^{-r(T-t)} E^Q \left[\max\left\{x_1 - S_T, 0\right\} + \max\left\{S_T - x_2, 0\right\} \,|\, \mathcal{F}_t \right]$$
$$= p(t, S_t, x_1, T, r, \sigma) + c(t, S_t, x_2, T, r, \sigma)$$

Here *c*(*t, s, K, T, r, σ*) denotes the standard Black-Scholes price at time *t* of a European call option with exercise price *K* and expiry date *T*, when the current price of the underlying is *s*, the interest rate is *r*, and the volatility of the underlying is *σ*. The price of the corresponding put option is denoted by *p*(*t, s, K, T, r, σ*).The value of *c*(*t, s, K, T, r, σ*) is given by the Black-Scholes formula, and using the put-call-parity

$$p(t, s, K, T, r, \sigma) = Ke^{-r(T-t)} + c(t, s, K, T, r, \sigma) - s$$

we can also find the value of *p*(*t, s, K, T, r, σ*) using the Black-Scholes formula. The price of the strangle is

$$\Pi_t = S_t \left(N\left[d_1(x_1)\right] + N\left[d_1(x_2)\right] - 1 \right) - e^{-r(T-t)} \left(x_1 N\left[d_2(x_1)\right] + x_2 N\left[d_2(x_2)\right] - x_1 \right)$$

where the expressions for d_1 and d_2 can be found in the formulas at the end of the problems, except that here the value within the parenthesis refers to which strike price K should be used.

Solution to problem 50

Denote the payoff function by Φ and note that

$$\Phi\,(S_T) = -a + \max\{S_T - x_1, 0\} - 2\max\{S_T - x_2, 0\} + \max\{S_T - x_3, 0\}$$

The butterfly strategy is constructed by buying a call option at the low strike x_1, selling two call options at the middle strike x_2 and by buying a third call option at the higher strike x_3.

The price of the contract is therefore

$$\begin{aligned}\Pi_t &= e^{-r(T-t)}E^Q\left[-a+\max\left\{S_T - x_1, 0\right\} - 2\max\left\{S_T - x_2, 0\right\} + \max\left\{S_T - x_3, 0\right\} \mid \mathcal{F}_t\right] \\ &= -a\cdot e^{-r(T-t)} + c(t, S_t, x_1, T, r, \sigma) - 2c(t, S_t, x_2, T, r, \sigma) + c(t, S_t, x_3, T, r, \sigma) \\ &= -a\cdot e^{-r(T-t)} + c(t, S_t, 0.95S_t, T, r, \sigma) - 2c(t, S_t, S_t, T, r, \sigma) + c(t, S_t, 1.05S_t, T, r, \sigma)\end{aligned}$$

Here $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is a. If we want the price of the claim to be zero a should be chosen as

$$a = e^{r(T-t)}\left[c(t, S_t, 0.95S_t, T, r, \sigma) - 2c(t, S_t, S_t, T, r, \sigma) + c(t, S_t, 1.05S_t, T, r, \sigma)\right]$$

where $c(t, s, K, T, r, \sigma)$ is given by the Black-Scholes formula.

Solution to problem 51

One way to show that there is arbitrage in the model is to show that there does not exist a martingale measure for the model. Since the filtration is the natural filtration generated by the Wiener process W, we only have to consider Girsanov transformations. Define a Girsanov transformation by

$$dQ = L(t)dP, \quad \text{on } \mathcal{F}_t$$

where

$$\begin{cases} dL_t = L_t g_t dW_t \\ L_0 = 1 \end{cases}$$

From Girsanov theorem we have that

$$dW_t = g_t dt + dV_t$$

where *V* is a *Q*-Wiener process. Thus, the *Q*-dynamics of *X* and *Y* is given by

$$\begin{cases} dX_t = (\alpha + \sigma g_t) X_t dt + \sigma X_t dV_t \\ dY_t = (\beta + \delta g_t) Y_t dt + \delta Y_t dV_t \end{cases}$$

In order for *Q* to be a risk neutral martingale measure, both *X* and *Y* have to have drifts equal to the short rate *r*. Thus we need *g* to satisfy

$$g_t = \frac{r-\alpha}{\sigma}, \text{ and } g_t = \frac{r-\beta}{\delta}$$

which require

$$\frac{r-\alpha}{\sigma} = \frac{r-\beta}{\delta}.$$

However, this cannot hold due to

$$r \neq \frac{\delta\alpha - \sigma\beta}{\delta - \sigma}.$$

This means that we cannot find a martingale measure for the model and thus, it is not free from arbitrage.

Solution to problem 52

Itô's formula gives the following *P*-dynamics for $Z^0(t) = B(t)/X(t)$ and $Z^1(t) = Y(t)/X(t)$

$$
\begin{aligned}
dZ_t^0 &= \frac{\partial Z_t^0}{\partial B_t} dB_t + \frac{\partial Z_t^0}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2 Z_t^0}{\partial X_t^{\,2}} \left(dX_t\right)^2 \\
&= \frac{1}{X_t} dB_t - \frac{B_t}{X_t^2} dX_t + \frac{1}{2} 2 \frac{B_t}{X_t^3} \left(dX_t\right)^2 = \left(r - \alpha + \sigma^2\right) Z_t^0 dt - \sigma Z_t^0 dW_t \\
dZ_t^1 &= \frac{\partial Z_t^1}{\partial Y_t} dY_t + \frac{1}{2}\frac{\partial^2 Z_t^1}{\partial Y_t^2}\left(dY_t\right)^2 + \frac{\partial Z_t^1}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2 Z_t^1}{\partial X_t^{\,2}}\left(dX_t\right)^2 + \frac{\partial^2 Z_t^1}{\partial X_t \partial Y_t}\left(dX_t dY_t\right) \\
&= -\frac{Y_t}{X_t^2} dX_t + \frac{1}{X_t} dY_t + \frac{1}{2} 2 \frac{Y_t}{X_t^3}\left(dX_t\right)^2 - \frac{1}{X_t^2} dX_t dY_t \\
&= \left(\beta - \alpha + \sigma^2 - \sigma\delta\right) Z_t^1 dt - \sigma Z_t^1 dW + \delta Z_t^1 dV_t
\end{aligned}
$$

Define a Girsanov transformation by

$$dQ^X = L(t) dP, \quad \text{on } \mathcal{F}_t$$

where

$$\begin{cases} dL_t = L_t g_t dW_t + L_t h_t dV_t \\ L_0 = 1 \end{cases}$$

From Girsanov theorem we have that

$$
\begin{aligned}
dW_t &= g_t dt + dW_t^X \\
dV_t &= h_t dt + dV_t^X
\end{aligned}
$$

where W^X and V^X are two independent Q^X-Wiener processes. Thus the Q^X dynamics of Z^0 and Z^1 are given by

$$\begin{cases} dZ_t^0 = \left(r - \alpha + \sigma^2 - \sigma g_t\right) Z_t^0 dt - \sigma Z_t^0 dW_t^X \\ dZ_t^1 = \left(\beta - \alpha + \sigma^2 - \sigma\delta - \sigma g_t + \delta h_t\right) Z_t^1 dt - \sigma Z_t^1 dW_t^X + \delta Z_t^1 dV_t^X \end{cases}$$

In order for Z^0 and Z^1 to be Q^X-martingale the drift terms must equals zero. This gives the following expressions for g and h

$$g = \frac{r - \alpha + \sigma^2}{\sigma}$$
$$h = \frac{r - \beta + \sigma\delta}{\delta}$$

An explicit expression for the likelihood process L is therefore

$$L_t = \exp\left\{-\frac{1}{2}(g+h)^2 t + gW_t + hV_t\right\}$$

Solution to problem 53

a) Under Q^d the process Z defined by

$$Z_t = \frac{X_t B_t^f}{B_t^d}$$

should be a martingale. Itô's formula gives the following dynamics for Z under P

$$dZ_t = \frac{X_t}{B_t^d} dB_t^f - \frac{X_t B_t^f}{\left(B_t^d\right)^2} dB_t^d + \frac{B_t^f}{B_t^d} dX_t = \left(r_f - r_d + \mu_X\right) Zdt + \sigma_X ZdW$$

Let

$$dQ^d = L_t dP \quad \text{on} \mathcal{F}_t$$

where

$$\begin{cases} dL = gLdW \\ L(0) = 1 \end{cases}$$

where

$$g = \frac{r_d - \mu_X - r_f}{\sigma_X}$$

By using the Girsanov Theorem we can then see that Z is a martingale under the new measure Q^d. Again using the Girsanov Theorem we find that the Q^d-dynamics of X are

$$dX = \mu_X Xdt + \sigma_X XdW = \mu_X Xdt + \sigma_X X\left(\frac{r_d - \mu_X - r_f}{\sigma_X}dt + dW^d\right)$$
$$= \left(r_d - r_f\right)Xdt + \sigma_X XdW^d$$

where W^d is a Q^d-Wiener process.

b) First of all we need the exchange rate process Y, which is used to convert domestic payoffs into foreign currency. This process is given by $Y = 1/X$. Using Ito's formula we obtain the following dynamics under Q^d

$$dY = -\frac{1}{X^2}dX + \frac{1}{2}\frac{2}{X^3}\left(dX\right)^2 = \left(r_f - r_d + \sigma_X^2\right)Ydt - \sigma_X YdW^d$$

Under Q^f the process ζ defined by

$$\varsigma_t = \frac{YB_t^d}{B_t^f}$$

should be a martingale. Itô's formula gives the following dynamics for ζ under Q^d

$$d\varsigma_t = \frac{Y_t}{B_t^f}dB_t^d - \frac{Y_t B_t^d}{\left(B_t^f\right)^2}dB_t^f + \frac{B_t^d}{B_t^f}dY_t = \sigma_X^2\varsigma_t dt - \sigma_X\varsigma_t dW^d$$

Let

$$dQ^f = LdQ^d \quad \text{on} \mathcal{F}_t$$

where

$$\begin{cases} dL = hLdW^d \\ L(0) = 1 \end{cases}$$

where

$$h = \sigma_X$$

By using the Girsanov Theorem we can then see that ζ is a martingale under the new measure Q^f.

c) In order for the two martingale measures to be equal (which they have to be if they are both to be equal to P) the likelihood process L must be identically equal to one (recall that $dQ^f = L_t dQ^d$ on F_t). Since we have that

$$L_t = \exp\left\{\int_0^t h(s)dW^d(s) - \frac{1}{2}\int_0^t h^2(s)ds\right\} = \exp\left\{\sigma_X W_t^d - \frac{1}{2}\sigma_X^2 t\right\}$$

we see that $L = 1$ requires $\sigma_x = 0$ and we get that the two measures are equal if the exchange rate is deterministic. In order for Q^d to be equal to P we find, using the same technique as above, that we must have $\mu_x = r_d - r_f$

$$L_t = \exp\left\{\int_0^t \frac{r_d - r_f - \mu_X}{\sigma_X} dW(s) - \frac{1}{2}\int_0^t \left(\frac{r_d - r_f - \mu_X}{\sigma_X}\right)^2 ds\right\}$$
$$= \exp\left\{\frac{r_d - r_f - \mu_X}{\sigma_X} W_t - \frac{1}{2}\left(\frac{r_d - r_f - \mu_X}{\sigma_X}\right)^2 t\right\}$$

Solution to problem 54

a) The financial market is arbitrage-free iff there exist a equivalent martingale measure Q with

$$E^Q\left[\frac{S(T)}{S_1(T)}\right] = \frac{S(0)}{S_1(0)}$$

b) We discount using the first security as numéraire. In order to find the equivalent martingale measure, we have to solve the following equations

$$\begin{pmatrix} \frac{S_1(1,\omega_1)}{S_1(1,\omega_1)} & \frac{S_1(1,\omega_2)}{S_1(1,\omega_2)} & \frac{S_1(1,\omega_3)}{S_1(1,\omega_3)} \\ \frac{S_2(1,\omega_1)}{S_1(1,\omega_1)} & \frac{S_2(1,\omega_2)}{S_1(1,\omega_2)} & \frac{S_4(1,\omega_3)}{S_1(1,\omega_3)} \\ \frac{S_3(1,\omega_1)}{S_1(1,\omega_1)} & \frac{S_3(1,\omega_2)}{S_1(1,\omega_2)} & \frac{S_3(1,\omega_3)}{S_1(1,\omega_3)} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \frac{S_1(0)}{S_1(0)} \\ \frac{S_2(0)}{S_1(0)} \\ \frac{S_3(0)}{S_1(0)} \end{pmatrix}$$

Using the values given above yields

$$\begin{pmatrix} 1 & 1 & 1 \\ 1.4 & 1.8 & 1.3 \\ 1.6 & 2 & 1.3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \\ \frac{\alpha}{100} \end{pmatrix}$$

This can be transformed to

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{17}{2} - \frac{\alpha}{20} \end{pmatrix}$$

Where we have $p_2 = (1 + p_3)/4$ and therefore $p_1 = 1 - 1/4 - 5p_3/4$. As $0 < p_3 < 1$ we get $p_3 < 3/5$. Then $0 < (170 - \alpha)/20 < 3/5$ and thus $158 < \alpha < 170$.

c) For $\alpha = 160$ the EMM is p = (1/8, 3.8, 1/2)
d) The price of the call is C(0) = $p_1 \cdot 20 + p_2 \cdot 60 = 25$.

Solution to problem 55

The price of the claim at time t ∈ £ [0, T1] is given by

$$\Pi_t[X] = e^{-r(T_1-t)} E^Q \left[\frac{1}{\Delta T} \int_{T_0}^{T_1} S_u du \,|\, F_t \right]$$

Change the order of integration to obtain

$$\Pi_t[X] = \frac{e^{-r(T_1-t)}}{\Delta T}\int_{T_0}^{T_1} E^Q\left[S_u \mid F_t\right]du$$

Now you can either use that the dynamics of S under Q are given by

$$dSt = rStdt + \sigma StdVt$$

where V is a Q-Wiener process to compute $E^Q[S_u|F_t]$, or you can use that S/B is a Q-martingale and that $B_t = e^{rt}$ in the following way

$$\Pi_t[X] = \frac{e^{-r(T_1-t)}}{\Delta T}\int_{T_0}^{T_1} E^Q\left[S_u \mid F_t\right]du = \frac{e^{-r(T_1-t)}}{\Delta T}\int_{T_0}^{T_1} e^{ru}E^Q\left[\frac{S_u}{e^{ru}} \mid F_t\right]du$$
$$= \frac{e^{-r(T_1-t)}}{\Delta T}\int_{T_0}^{T_1} e^{ru}\frac{S_t}{e^{rt}}du = \frac{e^{-rT_1}}{\Delta T}S_t\int_{T_0}^{T_1} e^{ru}du = \frac{e^{-rT_1}}{\Delta T}\frac{S_t}{r}\left[e^{rT_1} - e^{rT_0}\right]$$

which is the price of the claim at time t. With

$$S_T = S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T - W_t)\right\}$$

So

$$E^Q\left[S_u \mid F_t\right] = S_t \cdot E^Q\left[\exp\left\{r\tau + \sigma\sqrt{\tau}z\right\} \mid F_t\right]$$

i.e.

$$E^Q\left[S_u \mid F_t\right] = S_t \cdot \int_{-\infty}^{\infty} e^{r\cdot\tau+\sigma\cdot\sqrt{\tau}z}\varphi(z)dz = \frac{S_t \cdot e^{r\cdot\tau}}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\tau+\sigma\cdot\sqrt{\tau}z-z^2/2}dz$$
$$= \frac{S_t \cdot e^{r\cdot\tau}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(z-\sigma\cdot\sqrt{\tau}\right)^2/2}dz = S_t \cdot e^{r\cdot\tau}$$

and insert above, giving the same result. We can also use

$$\begin{cases} dS(t) = r\cdot S(t)\cdot dt + \sigma\cdot S(t)\cdot dW \\ S(T) = s \end{cases}$$

If we integrate this we will get

$$S(t)=s+r\int_0^t S(u)\cdot du+\sigma\int_0^t S(u)\cdot dW(u)$$

The price is given by

$$\Pi\left[X\mid F\right]=e^{-r(T_1-t)}E^Q\left[\frac{1}{\Delta T}\int_{T_0}^{T_1}S(u)du\right]=\frac{e^{-r(T_1-t)}}{\Delta T}\int_{T_0}^{T_1}E^Q\left[S(u)\right]du$$

We then calculate the expectation value of *S*(*T*):

$$E\left[S(T)\right]=S_t+r\int_0^T E\left[S(u)\right]\cdot du+0$$

Let *E*[*S*(*T*)] = *m* and take the derivative

$$\begin{cases}m(t)=r\cdot m(t)\\ m(0)=S_t\end{cases}$$

The solution is given by

$$m(t)=E\left[S(t)\right]=S_te^{rt}$$

This means that the price is given by

$$\Pi\left[X\mid F\right]=\frac{S_t\cdot e^{-rT_1}}{\Delta T}\int_{T_0}^{T_1}e^{ru}du=\frac{e^{-rT_1}}{\Delta T}\frac{S_t}{r}\left[e^{rT_1}-e^{rT_0}\right]$$

Solution to problem 56

Let $Z_t = 1/S_t$. Using Itô's formula we obtain the dynamic of *Z* under *Q* as

$$dZ=-\frac{1}{S^2}dS+\frac{1}{2}\frac{2}{S^3}\left(dS\right)^2=\left(\sigma^2-r\right)Zdt-\sigma ZdV$$

where *V* is a *Q*-Wiener process. Integrate and take conditional expectation to obtain

$$E^Q\left[Z_u \mid F_t\right] = Z_t + \left(\sigma^2 - r\right)\int_t^u E^Q\left[Z_s \mid F_t\right]ds$$

Let $m(u) = E[Z_u \mid F_t]$ and take the derivative w.r.t. u. This yields the following ODE for m

$$\begin{cases} m = \left(\sigma^2 - r\right)m \\ m(t) = Z_t \end{cases}$$

Solving the ODE we obtain

$$m(u) = E^Q\left[Z_u \mid F_t\right] = Z_t e^{\left(\sigma^2 - r\right)(u-t)}$$

The price at time $t \in [0, T_0)$ of "the inverse mean" is therefore given by

$$\begin{aligned}
\Pi(t,X) &= e^{-r(T_1-t)}E^Q\left[\int_{T_0}^{T_1}\frac{1}{S_u}du \mid F_t\right] = e^{-r(T_1-t)}\int_{T_0}^{T_1}E^Q\left[Z_u \mid F_t\right]du \\
&= e^{-r(T_1-t)}\int_{T_0}^{T_1}Z_t e^{\left(\sigma^2-r\right)(u-t)}du = e^{-r(T_1-t)}\frac{Z_t}{\sigma^2-r}\left[e^{\left(\sigma^2-r\right)(T_1-t)} - e^{\left(\sigma^2-r\right)(T_0-t)}\right] \\
&= e^{-r(T_1-t)}\frac{e^{-\left(\sigma^2-r\right)t}}{S_t\left(\sigma^2-r\right)}\left[e^{\left(\sigma^2-r\right)T_1} - e^{\left(\sigma^2-r\right)T_0}\right]
\end{aligned}$$

Solution to problem 57

A futures contract on X with time of delivery T is a financial asset with price process Π and dividend process D with the following properties

$$\begin{aligned}
&D(t) = F(t, T, X) \\
&F(T, T, X) = X \\
&\Pi(t) = 0 \text{ for } 0 \le t \le T.
\end{aligned}$$

Here $F(t, T, X)$ denotes the futures price process. Note that $F(t, T, X)$ is determined at time t. Recall that if Q is a martingale measure for this model, then the normalized gain process of any price-dividend pair $[\Pi, D]$ is a Q-martingale. Thus we have that

$$G^Z(t) = \frac{\Pi(t)}{B(t)} + \int_0^t \frac{1}{B(s)} dD(s)$$

is a Q-martingale. Now use that $\Pi(t) = 0$, and that $D(t) = F(t, T, X)$ along with the martingale representation theorem to obtain that

$$dG^Z(t) = \frac{dF(t,T,X)}{B(t)} = h(t)dV(t)$$

for some adapted process h and the Q-Wiener process V generating the filtration. From this we see that

$$dF(t, T, X) = B(t)h(t)dV(t)$$

which means that the futures price process is a Q-martingale. Using the martingale property of the futures price process and the boundary condition
$F(T, T, X) = X$, we have that the futures prices are given by

$$F(t, T, X) = E^Q[X \mid Ft]$$

Solution to problem 58

Recall that for this model the Q-dynamics of S are given by

$$dS(t) = (r - \delta)S(t)dt + \sigma S(t)dV(t)$$

where V denotes a Q-Wiener process. Integrating this we obtain

$$S(u) = S(t) + \int_t^u (r - \delta)S(\tau)d\tau + \int_t^u \sigma S(\tau)dV(\tau)$$

Now take the conditional expectation with respect to $\mathcal{F}_t$

$$E\left[S_u \mid \mathcal{F}_t\right] = S_t + \int_t^u (r - \delta)E\left[S_\tau \mid \mathcal{F}_t\right]d\tau + 0$$

Let $m_u = E[S_u \mid \mathcal{F}_t]$ and take derivatives with respect to u

$$\begin{cases} m_u = (r-\delta)m_u \\ m_t = S_t \end{cases}$$

Solving the ODE above we get

$$m_u = S_t e^{(r-\delta)(u-t)}$$

The futures price for this case is therefore

$$F(t,T,X) = S(t)e^{(r-\delta)(T-t)}$$

Solution to problem 59

(a) A Girsanov transformation with the Girsanov kernel

$$g_t = \frac{\alpha}{\sigma}\sqrt{X_t}$$

will change the dynamics of X in the desired way. The dynamics of the likelihood process L^{α} are given by

$$\begin{cases} dL_t^{\alpha} = \dfrac{\alpha}{\sigma}\sqrt{X_t}\,L_t^{\alpha}\,dW_t \\ L_0^{\alpha} = 1 \end{cases}$$

The likelihood process is thus given by

$$L(t) = \exp\left\{\int_0^t \frac{\alpha}{\sigma}\sqrt{X_s}\,dW_s - \frac{1}{2}\int_0^t \frac{\alpha^2}{\sigma^2}X_s ds\right\}$$

(b) You obtain the same estimate if you maximize the logarithm of the likelihood function. The maximum likelihood-estimate is thus given as the solution to the following problem

$$\max_{\alpha}\left\{\int_0^t \frac{\alpha}{\sigma}\sqrt{X_s}\,dW_s - \frac{1}{2}\int_0^t \frac{\alpha^2}{\sigma^2}X_s ds\right\}$$

Since we want the answer to be expressed in terms of X we write this as

$$\max_{\alpha}\left\{\frac{\alpha}{\sigma^2}\int_0^t dX_s - \frac{1}{2}\int_0^t \frac{\alpha^2}{\sigma^2} X_s ds\right\} = \max_{\alpha}\left\{\frac{\alpha}{\sigma^2}\left(X_t - X_0\right) - \frac{1}{2}\int_0^t \frac{\alpha^2}{\sigma^2} X_s ds\right\}$$

Since the objective function is concave in a the maximum will be obtained in a point where the derivative w.r.t. α is zero. This yields

$$\hat{\alpha}_t = \frac{X_t - X_0}{\int_0^t X_s ds}$$

Solution to problem 60

(a) Ito's formula gives us

$$dF = \left(F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right)dt + \sigma F_x dW$$

By definition, and using Itô's formula on F_x, we have

$$\begin{aligned} F_x \quad dX &= F_x dX + \frac{1}{2} dF_x dX = F_x\left(\mu dt + \sigma dW\right) + \\ &+ \frac{1}{2}\left\{F_{xt} dt + \mu F_{xx} dt + \frac{1}{2}\sigma^2 F_{xxx} dt + \sigma F_{xx} dW\right\}\left(\mu dt + \sigma dW\right) = \\ &= \left\{\mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right\} dt + \sigma F_x dW \end{aligned}$$

Thus we see that

$$F_t dt + F_x \quad dX = \left\{F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right\} dt + \sigma F_x dW = dF$$

(b) Using the formula we have

$$\int_0^t X(s)\ dX(s) = \int_0^t X(s)dX(s) + \frac{1}{2}\int_0^t \left(dX(s)\right)^2 =$$

$$\left[\frac{1}{2}X^2(s) - \frac{t}{2}\right]_0^t + \frac{1}{2}\int_0^t dt = \frac{1}{2}\left[X^2(t) - X^2(0)\right]$$

Questions

1. What is an OTC derivative?
2. How, when and why are new option series created.
3. What is liquidity risk?
4. What kind of market risks exist?
5. What is the difference between credit- and counterparty risk.
6. How is counterparty risk calculated?
7. What is model risk and why is model risk important?
8. What is settlement risk?
9. What is Herstatt risk?
10. Explain the purpose of a clearing house.
11. In model validation, how can you decide wat model is the best one in a certain market?
12. How can you increase the accuracy in the Binomial models and why do the oscillations disappear?
13. Explain how and when we can use Richardson extrapolation in the binomial model.
14. How can you calculate the Greeks in the binomial model and why do you need to make two trees to calculate vega and rho?
15. How can you calculate the probability to reach the strike price in the binomial model?
16. Give the boundary condition for a call- and a put option in the binomial model.
17. Why do you need to calculate the risk-free (martingale) probabilities to use the Binomial model?
18. How can you estimate the up and down factors in the Binomial model?

19. What is a replicating portfolio?
20. What is a headge portfolio?
21. What is a contingent claim?
22. What do we mean by long and short positions?
23. What is the difference in risk by going short or long in a call option?
24. Are binomial models Markov? Why?
25. What is the Meta theorem?
26. What is a martingale?
27. How can you estimate volatility and what is implied volatility.
28. What do we mean by a complete market?
29. What is the Hopscotch method and what makes is special?
30. Explain the three components in Value-at-Risk.
31. Why does management like VaR?
32. How can you calculate VaR?
33. Define expected shortfall.
34. Under what conditions can you have an optimal early exercise of an option?
35. What are the properties of a Weiner process?
36. What is the definition of a Radon-Nikodym derivative?
37. When do we use Radon-Nikodym derivatives?
38. What is the Itô lemma and why do we need the lemma?
39. What is unique with parabolic partial differential equations?
40. What is Feynmann-Kač representation and how to use it?
41. Why do we need to use Itô integration when we are dealing with finance?
42. Why can the Black-Scholes formula be applied for American call options but not on American puts?
43. What are the basic assumptions in the Black-Scholes model?
44. What’s de difference between the Black-Scholes model and the Black model?