Options on dividend paying equities

If we assume that the asset will pay a continuous dividend yield, *D*, then in time *t* the asset receives an amount *DSdt*. Then:

$$\frac{\partial F}{\partial t} + (r - D) S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0$$

Currency options

Options on currency are handled in exact the same way. In holding the foreign currencies we receive interest at the foreign rate on interest r_f. The Black-Scholes is now given by:

$$\frac{\partial F}{\partial t} + \left(r - r_f\right) S \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0$$

Options on commodities

Commodity options are also handled in the same way. In holding a commodity we have a carry of cost, *cc*, and the Black-Scholes is given by:

$$\frac{\partial F}{\partial t} + (r + cc) S \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0$$

Assumptions in the Black-Scholes world

- The underlying is a log normally distributed stochastic variable
- The volatility of the underlying is constant
- Interest rates are constant
- There are no transaction costs in any capital markets
- Borrowing and lending can be done at constant interest rate
- All strike-prices are tradable and liquid.
- There is continuous trading in all instruments.

Black-Scholes Formula

$$P_{call} = Se^{-qT}N(d_1) - Ke^{-rT}N\left(d_2\right)$$

$$P_{put} = Ke^{-rT}N(-d_2) - Se^{-qT}N\left(-d_1\right)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma \cdot \sqrt{T}}, \qquad d_2 = d_1 - \sigma \cdot \sqrt{T}$$

$$N(x) = \frac{1}{\sqrt{2 \cdot \pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

$$N'(x) = \frac{1}{\sqrt{2 \cdot \pi}} e^{-y^2/2}$$

Normal distribution

$$N(x) = 1 - N'(x)\left(a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5\right); \quad \text{för } x \geq 0$$

$$N(x) = 1 - N(-x); \quad \text{för } x < 0$$

$$y = \frac{1}{1 + \gamma x}$$

$$\gamma = 0.2316419$$

$$a_1 = 0.319381530$$

$$a_2 = -0.356563782$$

$$a_3 = 1.781477937$$

$$a_4 = -1.821255978$$

$$a_5 = 1.330274429$$

Black-Scholes and time-dependent parameters

We can generalize Black-Scholes by letting the parameters become time-dependent. Suppose that the interest rate and volatility are time dependent ($r \to r(t)$, $\sigma \to \sigma(t)$). We also introduce a time dependent dividend yield ($q \to q(t)$). The Black-Scholes equation then takes the form:

$$\frac{\partial F(t,S)}{\partial t} + \big(r(t) - q(t)\big) S(t) \frac{\partial F(t,S)}{\partial S} + \frac{1}{2}\sigma^2(t) S^2(t) \frac{\partial^2 F(t,S)}{\partial S^2} - r(t) F(t,S) = 0$$

If we now introduce the new variables:

$$S(t) = S(t) e^{\alpha(t)}, \quad F(t,S) = F(t,S) e^{\beta(t)}, \quad \tau = \gamma(t)$$

to eliminate the time dependence we get the new equation:

$$\dot{\gamma}(t) \frac{\partial \bar{F}}{\partial \tau} + \big(r(t) - q(t) + \dot{\alpha}(t)\big) \bar{S}(t) \frac{\partial \bar{F}}{\partial \bar{S}} + \frac{1}{2}\sigma^2(t) \bar{S}^2(t) \frac{\partial^2 \bar{F}}{\partial \bar{S}^2} - \big(r(t) + \dot{\beta}(t)\big) \bar{F} = 0$$

where the dots denotes time derivatives.

Black-Scholes and time-dependent parameters

To eliminate the time dependent terms we now choose:

$$\alpha(t) = \int_t^T \left(r(\tau) - q(\tau)\right) d\tau,$$

$$\beta(t) = \int_t^T r(\tau) d\tau$$

$$\gamma(t) = \int_t^T \sigma^2(\tau) d\tau$$

The formula for a European call option with time-dependent parameters

$$\Pi = S \exp\left\{-\int_t^T q(\tau) d\tau\right\} N\left(d_1\right) - K \exp\left\{-\int_t^T r(\tau) d\tau\right\} N\left(d_2\right)$$

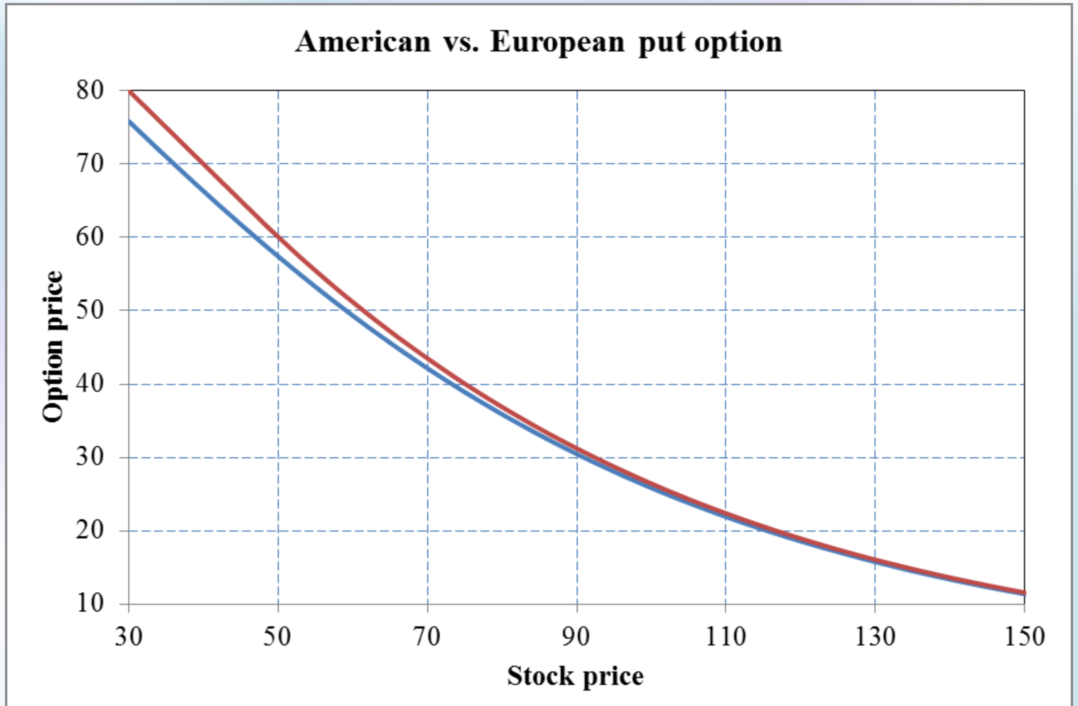
Black-Scholes and time-dependent parameters

$$d_1 = \frac{\ln\left\{\frac{S}{K}\right\} + \int_t^T \left(r(\tau) - q(\tau)\right) d\tau + \frac{1}{2}\int_t^T \sigma^2(\tau) d\tau}{\sqrt{\int_t^T \sigma^2(\tau) d\tau}}$$

$$d_2 = \frac{\ln\left\{\frac{S}{K}\right\} + \int_t^T \left(r(\tau) - q(\tau)\right) d\tau - \frac{1}{2}\int_t^T \sigma^2(\tau) d\tau}{\sqrt{\int_t^T \sigma^2(\tau) d\tau}}$$

American versus European options

- The boundary condition do not have a well defined at maturity. We have a “floating” boundary condition.
- We cannot find a put-call parity for an American option.
- There exist a number of approximations, but any general closed form solution does not exist.
- The possibility of early exercise gives the American option a higher price than a European. An **early exercise premium**.

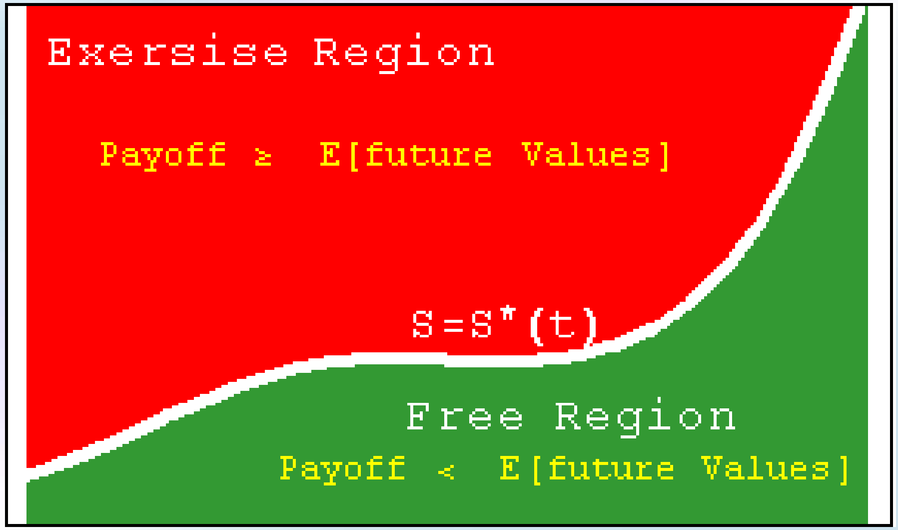


American versus European options

The contract holder will ideally only exercise prior to expiry if the present *payoff* at *t* exceeds the discounted expectation of the possible *future* values.

At all *t* there will be a region of values of *S* where it is best to exercise and a region where it is best to keep the option.

There will also be a value *S*(t)* which defines the *optimal exercise boundary*.



Currency options and Garman-Kohlhagen

In 1983 Garman and Kohlhagen extended the Black-Scholes model to cope with the presence of two interest rates (one for each currency). These also called foreign exchange option or FX-options.

Suppose that r_d is the risk-free interest rate to expiry of the domestic and r_f in the foreign currency The strike and current spot be quoted in terms of "units of domestic per unit of foreign currency".

We consider the model *geometric Brownian motion*:

$$dS_t = \left(r_d - r_f\right) S_t + \sigma S_t dW_t$$

In the case of EUR-USD, a spot of 1.2000, means that the price of one EUR is 1.2000 USD. The notion of *foreign* and *domestic* does refer only to the quotation convention. Applying Itô and we get:

$$S_t = S_t \exp\left\{\left(r_d - r_f - \frac{1}{2}\sigma^2\right)t + \sigma \cdot W_t\right\}$$

Currency options and Garman-Kohlhagen

This shows that ln S_t is normal with mean ln $S_0 + (r_d - r_f - ½\sigma^2)t$ and variance $\sigma^2 t$. The payoff for a vanilla option (European put or call) is given by: $\Phi = [\varepsilon(S_T - K)]^+$ where ε, = +1 for a call and −1 for a put.

The value $V(t, x)$ and can be computed as

$$V(x,K,t,T,\sigma,r_d,r_f,\varepsilon) = e^{-r_d \cdot (T-t)} E[F \mid \mathcal{F}_t]$$

Then the domestic currency value of a call option into the foreign currency is:

$$V_0 = \varepsilon \cdot e^{-r_d \cdot T_d} \left\{ f \cdot N(\varepsilon \cdot d_1) - K \cdot N(\varepsilon \cdot d_2) \right\}$$

$$d_1 = \frac{\ln\{S_0 / K\} + (r_d - r_f) \cdot T_d + \sigma^2 \cdot T_e / 2}{\sigma \cdot \sqrt{T_e}}, \quad d_2 = d_1 - \sigma \cdot \sqrt{T_e}$$

where f is the forward price of the underlying:

$$E[S_T \mid S_t = x] = x \cdot \exp\{(r_d - r_f)T_d\}$$

Currency options and Garman-Kohlhagen

x	spot FX rate domestic units per foreign currency, i.e. the price of the underlying.
K	strike using the same quotation as the spot rate
T_e	time from today until expiry of the option
T_d	time from spot until delivery of the option
r_d	domestic interest rate corresponding with period T_d
r_f	foreign interest rate corresponding with period T_d
σ	volatility corresponding with strike K and period T_e

The *forward price* f is the strike, which makes the time zero value of the *forward contract* $V = S_T - f$ equal to zero.

The situation $r_d > r_f$ is called *contango*, and the situation $r_d < r_f$ is called *backwardation*.

Currency options and Garman-Kohlhagen

The Black-Scholes delta also called spot delta of the option is equal to

$$\Delta_{BS} = \frac{\partial V}{\partial x} = \varepsilon \cdot e^{-r_f \cdot T_d} N(\varepsilon \cdot d_1)$$

The dual delta is defined by

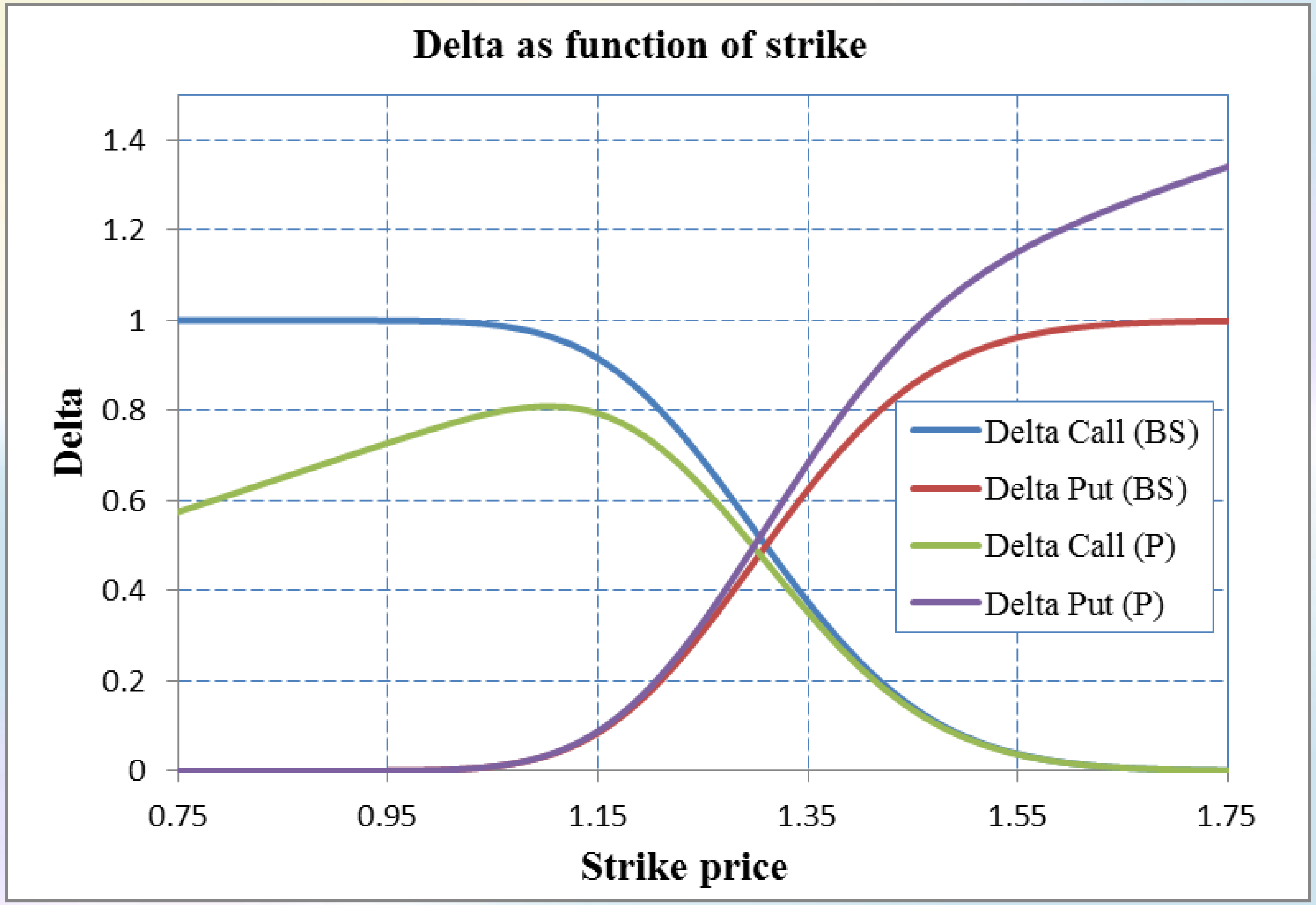
$$\Delta_{BS}^{dual} = -\varepsilon \cdot e^{-r_d \cdot T_d} N(\varepsilon \cdot d_2)$$

In all currency markets, except the EuroDollar market, the premium in the foreign currency is included in the delta. This "premium-included" delta has to be calculated as follows:

$$\Delta_p = \Delta_{BS} - \frac{V}{x} = \varepsilon \cdot \frac{K}{x} e^{-r_d \cdot T_d} N(\varepsilon \cdot d_1)$$

The logic of this delta can be illustrated with a bank that sells a call on the foreign currency. This option can be delta hedged in the foreign currency. The bank only have to buy an amount equal to the premium-included delta when it receives the premium in foreign currency.

Currency options and Garman-Kohlhagen



Black-Scholes and premium-included delta as function of strike.

Currency options and Garman-Kohlhagen

It can be observed from the above formula that the premium-included delta for a call is not strictly decreasing in strike like the Black-Scholes call delta. Therefore, a premium-included call delta can correspond to two possible strike prices.

For emerging markets (EM) and for maturities of more than two years, it is usual for forward delta's to be quoted. These are defined as follows

$$\Delta^F{}_{BS} = e^{r_f \cdot T_d} \Delta_{BS} \quad \text{and}$$

$$\Delta^F{}_{P} = e^{r_f \cdot T_d} \Delta_{P}$$

The quotation of FX-Options is a constantly confusing issue, so let us clarify this here. The exchange rate means how much of the *domestic* currency is needed to buy one unit of *foreign* currency. For example, if we take EUR/USD as an exchange rate, then the default quotation is EUR-USD, where USD is the domestic currency and EUR the foreign currency. The term *domestic* is in no way related to the location of the trader or any country. It merely means the *numeraire* currency

Currency options and Garman-Kohlhagen

The terms *domestic*, *numeraire* or *base currency* are synonyms as are *foreign* and *underlying*. EUR/USD can also be quoted in either EUR-USD, which then means how many USD are needed to buy one EUR, or in USD-EUR, which then means how many EUR are needed to buy one USD. There is certain market standard quotations listed in table below:

Currency pair	Default quotation	Sample quote
GBP/USD	GPB-USD	1.8000
GBP/CHF	GBP-CHF	2.2500
EUR/USD	EUR-USD	1.2000
EUR/GBP	EUR-GBP	0.6900
EUR/JPY	EUR-JPY	135.00
EUR/CHF	EUR-CHF	1.5500
USD/JPY	USD-JPY	108.00
USD/CHF	USD-CHF	1.2800

About the Volatility

The only unobserved input on the market is the volatility. But we can calculate the implied volatility for a certain price. If all Black-Scholes assumptions would hold, the implied volatility would be the same for all European vanilla options on a specific underlying FX rate.

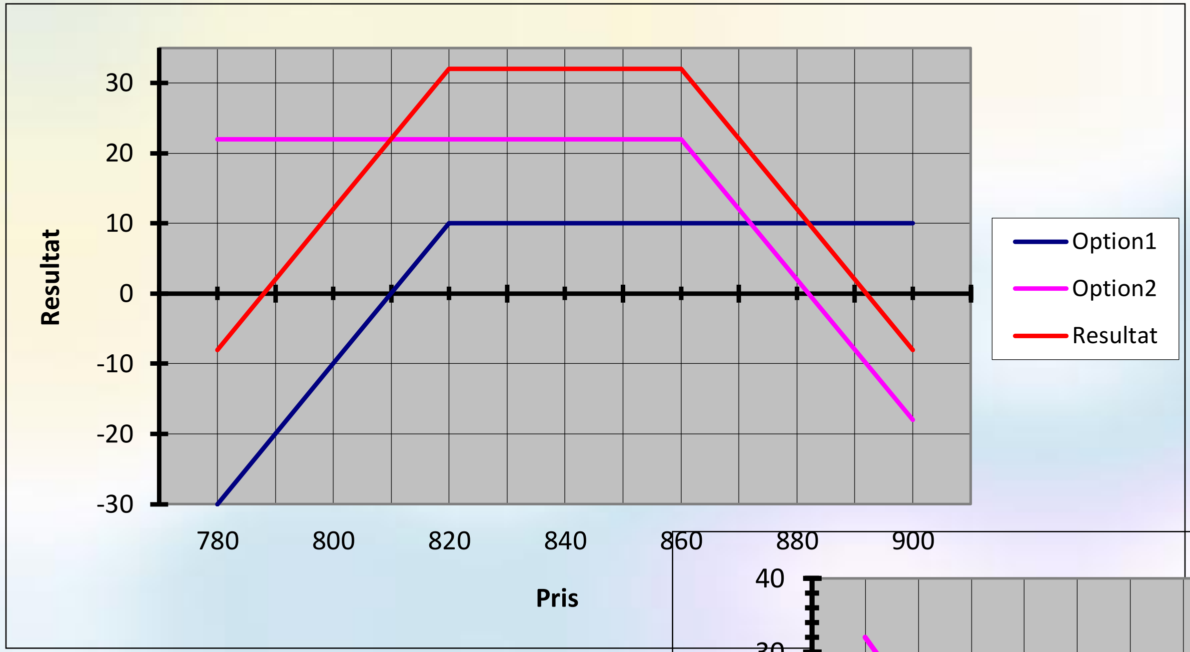
In reality we will find different implied volatilities for different strikes and maturities. Black-Scholes model effectively acts as a quotation convention due to the price <--> volatility relation.

In the table we show how in the FX market implied volatilities are quoted:

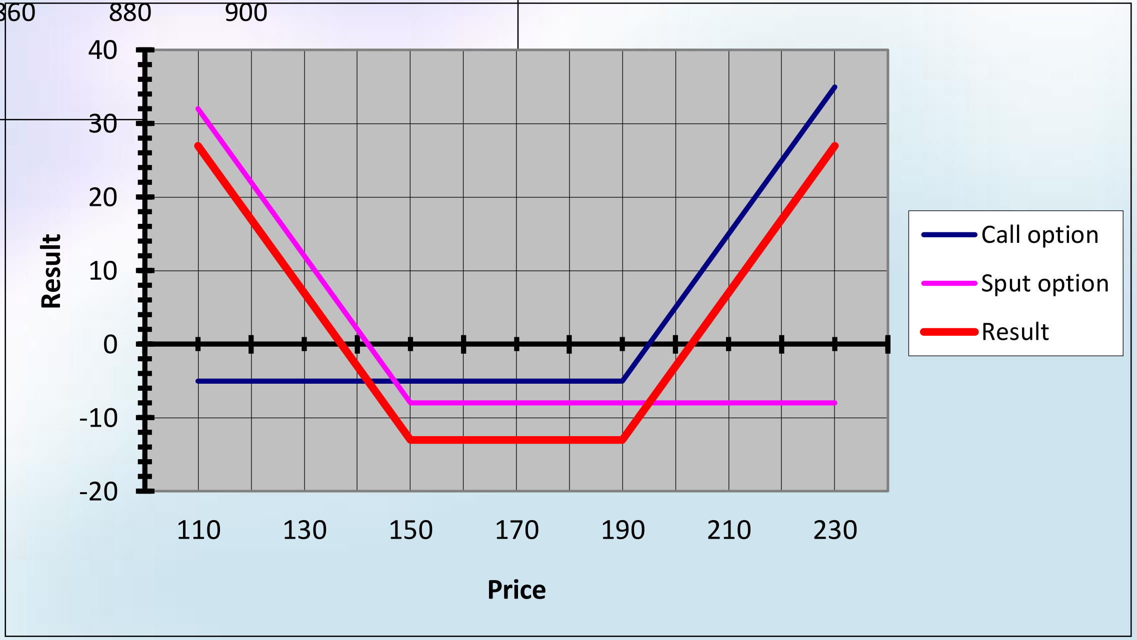
GBP/USD Spot 1.6459

	Vols	25 Delta Strangles	25 Delta Risk Revs
1W	8.08		
1M	8.18	0.19	0.59
3M	8.26	0.21	0.52
6M	8.38		
1Y	8.48	0.22	0.41
2Y	8.67		

Strangles



30
20
10
0
-10
-20
-30
Resultat
780
800
820
840
860
880
900
Pris
Option1
Option2
Resultat



40
30
20
10
0
-10
-20
Result
110
130
150
170
190
210
230
Price
Call option
Sput option
Result

About the Volatility

We saw above "Vols" used for At-the-Money (ATM) options of various maturities. Furthermore, quotation Strangles (STR) and Risk Reversals (RR). A strangle is a long position in an Out-of-the-Money (OTM) call and an OTM put. A strangle is a bet on a large move of the underlying either upwards or downwards.

A risk reversal is a combination of a long OTM call and a short OTM put.

Delta is the sensitivity of the option to the spot FX rate [0%,100%] of the notional. An ATM option have delta around 50%. A 25-delta call (put) corresponds to option with a strike above (below) the strike of an ATM option.

A 25-delta RR quote is the difference between the volatility of a 25-delta call and a 25-delta put. A 25-delta STR is equal to the average volatility of a 25-delta call and put minus the ATM volatility. Therefore, the volatility of a 25-delta call and put can be obtained from these quotes as follows:

About the Volatility

$$\sigma_{C,25} = \sigma_{ATM} + STR_{25} + \frac{1}{2} RR_{25}$$

$$\sigma_{P,25} = \sigma_{ATM} + STR_{25} - \frac{1}{2} RR_{25}$$

To derive the value for European vanilla options for other delta’s one needs to interpolate between and extrapolate outside the available quotes.

$$\sigma(t) = \sigma(t_1) + \frac{\sqrt{t} - \sqrt{t_1}}{\sqrt{t_2} - \sqrt{t_t}} \left(\sigma(t_2) - \sigma(t_1) \right); \qquad t_1 \le t \le t_2$$

Another interpolation is the linear total variance method for the implied volatility

$$\sigma_t = \frac{1}{\sqrt{t}} \left[T_1 \sigma_1^2 + \left(\frac{t - T_1}{T_2 - T_1} \right) \left(T_2 \sigma_2^2 - T_1 \sigma_1^2 \right) \right]^{1/2}$$

Volatility in terms of delta

In the FX market implied volatilities are quoted in terms of delta. (Remember the various definitions!).

The mapping $\sigma \to \Delta = \varepsilon \exp\{-r_f(T - t)\} N(\varepsilon d_1)$ is not one-to-one. The two solutions are given by:

$$\sigma_{\pm} = \frac{1}{\sqrt{T-t}} \left\{ \varepsilon \cdot N^{-1}(\varepsilon \cdot \Delta \cdot e^{r_f (T-t)}) \pm \sqrt{\left[N^{-1}(\varepsilon \cdot \Delta \cdot e^{r_f (T-t)}) \right]^2 - \sigma\sqrt{T-t} \cdot (d_1 - d_2)} \right\}$$

The determination of the volatility and the delta for a given strike is an iterative process:

1. Choose σ_0 = at-the-money volatility from the volatility matrix.
2. Calculate $\Delta_{n+1} = \Delta(\text{Call}(K, \sigma_n))$.
3. Take $\sigma_{n+1} = \sigma(\Delta_{n+1})$ from the volatility matrix, possibly via a suitable interpolation.
4. If $|\sigma_{n+1} - \sigma_n| < \delta$, then quit, otherwise continue with step 2.

Some trading floor language

We call one million a *buck*, one billion a *yard*. This is because a billion is called ‘milliarde’ in French, German and other languages. For the British Pound one million is also often called a *quid*.

Certain currency pairs have names. For instance, GBP/ USD is called *cable*, because the exchange rate information used to be sent through a cable in the. EUR/JPY is called the *cross*, because it is the cross rate of the more liquidly traded USD/JPY and EUR/USD.

Certain currencies also have names, e.g. the New Zealand Dollar NZD is called a *kiwi*, the Australian Dollar AUD is called *Aussie*, and the Scandinavian currencies DKR, NOK and SEK are called *Scandies*.

Exchange rates are generally quoted up to five relevant figures, e.g. in EUR-USD we could observe a quote of 1.2375. The last digit ‘5’ is called the *pip*, the middle digit ‘3’ is called the *big figure*.

To make it clear, a rise of USD-JPY 108.25 by 20 pips will be 108.45 and a rise by 2 big figures will be 110.25.

Roll, Geske & Whaley

An American call option can be considered to be a series of call options which expire at the ex-dividend dates, and this case becomes a compound option or (an option on an option) with a closed-form solution as follows:

$$C_D = \left(S_0 - D_1 e^{-rt_1}\right) N\left(b_1\right) + \left(S_0 - D_1 e^{-rt_1}\right) M\left(a_1, -b_1; -\sqrt{t_1/T}\right)$$
$$- X e^{-rT} M\left(a_2, -b_2; -\sqrt{t_1/T}\right) - \left(X - D_1\right) e^{-rt_1} N\left(b_2\right),$$

$$a_1 = \frac{\log\left[\left(S_0 - D_1 e^{-rt_1}\right)/X\right] + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}},$$

$$a_2 = a_1 - \sigma\sqrt{T},$$

$$b_1 = \frac{\log\left[\left(S_0 - D_1 e^{-rt_1}\right)/S^*\right] + \left(r + \sigma^2/2\right)t_1}{\sigma\sqrt{t_1}},$$

$$b_2 = b_1 - \sigma\sqrt{t_1},$$

where $M(a, b, .\ )$is the bivariate cumulative normal distribution function and S^* is the critical stock price for which the following equation is satisfied:

Roll, Geske & Whaley

where $c(S^*) = S^* + D_1 - X$, is the price given by Black-Scholes. The critical stock price can be solved iteratively via an numerical method.

Barone-Adesi, Whaley

BAW gave a quadratic approximation to price American Options, and the pricing of the option is essentially a European option with adjusted for an early exercise premium. See Lecture notes.

Bjerksund, Stensland

The Bjerksund-Stensland approximation assumes that the exercise is initiated to a corresponding 'flat' boundary, making use of a trigger price. This approximation is computational inexpensive and the method is fast, with evidence indicating that the approximation may be more accurate in pricing long dated options than the Barone-Adesi & Whaley model.

Martingale Representation

Theorem: Let W be a Wiener process on [0, 1] and M a martingale such as:

M is $\mathcal{F}_t^W$-adapted

$E[M^2(t)] < \infty \ \forall\, t \in [0, T]$

Then, there exist a $\mathcal{F}_t^W$-adapted process g such as:

(1) $$E\left[\int_0^T g(s)dW(s)\right] < \infty$$

(2) $$M(T) = M(0) + \int_0^T g(s)dW(s)$$

or

$$dM(t) = g(t)dW(t)$$

This is an abstract existence result that guarantees the existence of the process $g(t)$ but it does not tell us how the process look likes.

Girsanov Transformation

Consider a stochastic binomial process *S* with p = 0.75 and q = 0.25. We ask ourselves if this is a fair game. Obviously is this not the case and for that reason is the process *S* in the continuous limit not a martingale. However, in the continuous limit we can model the stochastic process with:

$$dS = \mu dt + \sigma dX$$

$$\mu = E[S] = (+1)\cdot p + (-1)\cdot q = 0.5$$

$$\sigma^2 = E[S^2] - \left(E[S]\right)^2 = (+1)^2 \cdot p + (-1)^2 \cdot q - (p-q)^2$$

$$= 1 - (p-q)^2 = (p+q)^2 - (p-q)^2 = 4pq = 0.75$$

With known μ and σ we can imitate *S* with a fair game via a normal distributed process *dX*. But, for this a transformation is needed. Therefore, let *dW* be another normal process such as:

$$dX = dW + \gamma dt$$

Girsanov Transformation

Obvious, the process dW is not a fair game with respect to the process dX, but with a shift γ the process will be a fair game in another coordinate system. For each unique outcome in dX there exists a unique outcome in dW and vice versa. We can write:

$$dS = \mu dt + \sigma\left(dW + \gamma dt\right)$$

If we choose $\gamma = -(\mu/\sigma)$ we get $dS = \sigma dW$. This means that we have removed the drift by a transformation so that dS is a martingale under a new probability measure. Now, dS is a fair game with respect to the process following the deterministic evolution $(\mu/\sigma)t$. The same argument can be used for the log-normal process whereby dS/S becomes martingale with respect to dW:

$$dS = \sigma S dW$$

Girsanov Transformation

Now, dS is a fair game with respect to the process following the deterministic evolution $e^{(\mu/\sigma)t}$. Pure financially, where $S(t)$ represents the value on a security at the time t, dS is not a fair game. The reason is that μ/σ is the expected pay-off with respect to the risk we take. But, if we discount with the risk free interest rate we will get a fair game. Therefore we have to choose:

$$\gamma = \frac{\mu - r}{\sigma}$$

This is called the **market price of [volatility] risk** or **the sharp quote** and is interpreted as the minimum extra pay-off needed to take the extra risk (in volatility).

Girsanov Theorem

Suppose that we are given a probability space $(\Omega, \mathcal{F}, P)$, where P is the market probability measure. Let X be a $(\mathcal{F}, P)$-Wiener process (a Brownian motion) and let $\mathcal{F}(t)$ be the filtration generated by this Wiener process. Also let L and g be as above ($g(t)$ is adapted to $\mathcal{F}(t)$). Furthermore suppose that $E[L(T)] = 1$ and define Q via $dQ = L(T)dP$ on $\mathcal{F}(t)$. Then:

$$W(t) = X(t) - \int_0^t g(s)ds$$

is a $(\mathcal{F}, Q)$-Wiener process (Brownian motion under Q).

<u>Interpretation</u>: X is a Q-Wiener process with the drift g.

An important point about Girsanov's theorem is that every equivalent measure is given by a drift change. This implies that in the Black-Scholes world there is only one equivalent risk-neutral measure. If this were not the case then there would be multiple arbitrage-free prices.

The reverse of the Girsanov theorem

Given $(\Omega, \mathcal{F}, P)$, X and suppose $Q << P$ on $\mathcal{F}_T^X$, then there exist a unique $\{\mathcal{F}_t^X\}$-adapted process g such as:

$$dQ(t) = L(t)dP(t)$$

where L is given by

$$dL(t) = g(t)L(t)dX(t), \quad L(0)=1$$

Remark! L is a Radon-Nikodym derivative.

Girsanov Theorem

Lemma: Let g be a $\mathcal{F}$-adapted process with

$$P\left(\int_0^T g^2(t)dt < \infty\right) = 1$$

Then

$$dL(t) = g(t)L(t)dW(t)$$
$$L(0)=1$$

have a unique solution $L > 0$:

$$L(t) = \exp\left\{\int_0^t g(s)dW(s) - \frac{1}{2}\int_0^t g^2(s)ds\right\}$$

Proof: Exercise (use Itô formula).

Black-Scholes world...

For a general market $Z = (S/B)$ with a unique martingale measure Q:

$$Q(A) = \int_{\Omega} Z dP(A)$$

We then have

$$\begin{aligned} dZ(t) &= \frac{\partial Z}{\partial S} dS + \frac{\partial Z}{\partial B} dB \\ &= \frac{1}{B(t)} \{\mu(t) S(t) dt + \sigma(t) S(t) dW(t)\} - \frac{S(t)}{B^2(t)} r(t) B(t) dt \\ &= \{\mu(t) - r(t)\} Z(t) dt + \sigma(t) Z(t) dW(t) \end{aligned}$$

With a Girsanov transformation $dW(t) = g(t)dt + dV(t)$ we get

Black-Scholes world...

$$dZ(t) = \{\mu(t) - r(t)\} Z(t)dt + \sigma(t)Z(t)\{g(t)dt + dV(t)\}$$
$$= \{\mu(t) - r(t) + \sigma(t)g(t)\} Z(t)dt + \sigma(t)Z(t)dV(t)$$

where

$$L(t) = \exp\left\{\int_0^t g(s)dW(s) - \frac{1}{2}\int_0^t g^2(s)ds\right\}$$

We call the function $g(t)$ the Girsanov kernel. $g(t)$ is given by

$$g(t) = \frac{r(t) - \mu(t)}{\sigma(t)}$$

We call the quotient $\dfrac{\mu(t) - r(t)}{\sigma(t)}$

the market price of risk or the sharp quote and denotes the excess rate return over the risk free rate of return on the market. $\mu(t)$ is then the expected return of the stock.

Black-Scholes World

Given a probability space (Ω, *F, P, W,* $\mathcal{F}$), chose a fix time T^* and let $\mathcal{F} = \{\mathcal{F}_t ; 0 \leq t \leq T^*\}$ be the natural filtration, $\mathcal{F}_t = \sigma\{W_s ; s \leq t\}$. Let's study a market (*B, S*):

$$\begin{cases} dB(t) = r \cdot B(t)dt \\ B(0) = 1 \end{cases} \qquad B(t) = e^{rt}$$

$$\begin{cases} dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) \\ S(0) = s \end{cases}$$

where r, α, and σ are constants, $\sigma > 0$.

Lemma: X is martingale if and only if $dX(t) = g(t)dW(t)$.

Now, we want to change probability measure so that Black-Scholes becomes martingale. We therefore introduce the *Z*-economy via $Z(t) = (Z^0(t), Z^1(t))$ so that:

Black-Scholes World

$$Z(t) = \frac{1}{B(t)}\left(B(t), S(t)\right) = \left(1, \frac{S(t)}{B(t)}\right)$$

is martingale. We want $S(t)/B(\mathrm{t}) \equiv e^{-rt}S(t)$ to be martingale. Therefore we need to find the dynamics under Q for Z^1.
Itô gives:

$$dZ^1(t) = \frac{\partial Z^1(t)}{\partial t}dt + \frac{\partial Z^1(t)}{\partial S}dS = -r \cdot e^{-rt}S(t)dt + e^{-rt}dS(t)$$
$$= (\alpha - r)Z^1(t)dt + \sigma Z^1(t)dW(t)$$

First, we have to find a Girsanov transformation so that $dZ^1(t)$ is martingale.

Let $dQ = L_T dP$ on $\mathcal{F}_{T^*}$, L_T is called a likelihood process.
We have:

Black-Scholes World

$$dL(t) = g(t)L(t)dW(t)$$
$$L(0) = 1$$

The theorem of Girsanov gives

$$dW(t) = g(t)dt + d\nu(t)$$

where $\nu(t)$ is a Q-Wiener process where the dynamics of Q for $Z^1(t)$ is given by:

$$dZ^1(t) = (\alpha - r)Z^1(t)dt + \sigma Z^1(t)\big(g(t)dt + d\nu(t)\big)$$
$$= (\alpha - r + \sigma g(t))Z^1(t)dt + \sigma Z^1(t)d\nu(t)$$

This dynamics is martingale if $(\alpha - r + \sigma g(t)) = 0$, i.e. if $g(t) = (r - \alpha)/\sigma$. The function $g(t)$ is called the Girsanov kernel. To sum up, under the martingale measure Q we have a Z-economy with the dynamic given by:

Black-Scholes World

$$\begin{cases} dZ^0(t) = 0 \\ dZ^1(t) = \sigma Z^1(t) d\nu(t) \end{cases}$$

$$Z^1(t) = e^{-rt} S(t) \quad \Rightarrow \quad dS(t) = r \cdot S(t) dt + \sigma S(t) d\nu(t)$$

Proof

$$S(t) = e^{rt} Z^1(t)$$

$$dS(t) = \frac{\partial S(t)}{\partial t} dt + \frac{\partial S(t)}{\partial Z^1(t)} dZ^1(t) = re^{rt} Z^1(t) dt + e^{rt} Z^1(t) \sigma d\nu(t)$$

$$= rS(t) dt + \sigma S(t) d\nu(t)$$

Which as before, is independent of α.

Black-Scholes World

If we define a likelihood process L via

$$dL_t = \frac{r-\alpha}{\sigma} L_t dW_t$$

we get

$$L(t) = \exp\left\{ \int_0^t \frac{r-\alpha}{\sigma} dW(s) - \frac{1}{2}\int_0^t \left(\frac{r-\alpha}{\sigma}\right)^2 ds \right\} = \exp\left\{ \frac{r-\alpha}{\sigma} W(t) - \frac{1}{2}\left(\frac{r-\alpha}{\sigma}\right)^2 t \right\}$$

and

$$dW_t = \frac{r-\alpha}{\sigma} dt + d\nu_t$$

When we change measure, and as above, with $B(t)$ as numeraire, we value all other instruments in terms of $B(t)$. In a general we can use any security S_i as the numeraire and value all other instruments with respect S_i. This means that we discount all securities with S_i as a *risk-free* asset (rate, rate of return).

Black-Scholes World

Definition: A probability measure Q is said to be a martingale measure if

- $Q \sim P$,
- Under Q the Z^1-dynamic is given by:

$$dZ_t^1 = \sigma Z_t^1 d\nu_t$$

where ν_t is a $(Q, \mathcal{F})$-wiener process.

Definition: A **portfolio strategy** is a stochastic process $h = (h^0, h^1)$ such as h is $\{\mathcal{F}_t\}$-adapted and integrable.

Black-Scholes World

Definition: The ***S*-value process** is given by:

$$V^s(t) = h^0(t)B(t) + h^1(t)S(t).$$

Definition: The ***Z*- value process** is given by:

$$V^z(t) = h^0(t) + h^1(t)Z^1(t).$$

Definition: ***S* is self-financed if**:

$$dV^s(t) = h^0(t)dB(t) + h^1(t)dS(t).$$

This can also be written as:

$$V^S(t) = h^0(0) \cdot V^S(0) + \int_0^T h^1(u)dS(u)$$

Black-Scholes World

Definition: ***Z* is self-financed if**: $dV^z(t) = h^1(t)dZ^1(t)$.

This can also be written as:

$$V^Z(t) = V^Z(0) + \int_0^t h^1(u)dZ(u)$$

Definition: A **contingent *T*-claim** is a stochastic variable X, which is $\mathcal{F}_t$-measurable and integrable.

Definition: ***S*-reachable**: $dV^s(t, h) = X$, *P almost true*.

Definition: ***Z*- reachable**: $dV^z(t, h) = X$, *Q almost true*.

Black-Scholes World

Definition: A **contingent *T*-claim** is a stochastic variable *X*, which is $\mathcal{F}_t$-measurable and integrable.

Definition: ***S*-reachable**: $dV^s(t, h) = X$, *P almost true*.

Definition: ***Z*- reachable**: $dV^z(t, h) = X$, *Q almost true*.

Theorem: Black-Scholes is free of arbitrage.

$$X = V^z(T,h) = \int_0^T h^1(t)dZ^1(t) = \int_0^T h^1(t)\cdot\sigma\cdot Z^1(t)d\nu(t)$$

$$E^Q[X] = E^Q\left[\int_0^T h^1(t)\cdot\sigma\cdot Z^1(t)d\nu(t)\right] = 0 \quad \text{since } \nu(t) \text{ is } Q\text{-martingale}$$

Since *P* and *Q* are equivalent measures ($P \sim Q$) this also holds in the *S* economy. To have an arbitrage possibility we must have $E^Q[X] > 0$ but we see from the above that $E^Q[X] = 0$.

Black-Scholes World

Theorem: Black-Scholes is complete, i.e. all contingent claims are reachable.

Proof: We study the Z-economy. For a given X show that there exist a portfolio h such as

$$V^z(t) = h^0(t) + h^1(t)Z^1(t) = X$$
$$dV^z(t) = h^1(t)dZ^1(t) = h^1(t)\cdot\sigma\cdot Z^1(t)d\nu(t)$$

on the probability measure Q where $\nu(t)$ is a Q-Weiner process. We know that if such a portfolio exists, the value process above is Q-martingale. If we use the martingale representation theorem

Black-Scholes World

$$M(t) = M(0) + \int_0^t g(s)dv(s) \quad \Rightarrow \quad dM(t) = g(t)dv(t)$$

and define a portfolio strategy as

$$\begin{cases} h^0(t) = M(t) - h^1(t)Z^1(t) \\ h^1(t) = \dfrac{g(t)}{\sigma \cdot Z^1(t)} \end{cases}$$

Then we get that

$$M(t) = h^0(t) + h^1(t)Z^1(t) = V^Z(t)$$

$$dV^Z(t) = h^1(t) \cdot dZ^1(t) = h^1(t) \cdot \sigma \cdot Z^1(t) \cdot dv(t) = g(t) \cdot dv(t) = dM(t)$$

Therefore, the portfolio is self-financed and X is Z-reachable which gives us that the model is complete.

Black-Scholes World

Theorem: In the Black-Scholes model, the martingale measure is given by Q where $Q \sim P$ and:

1.) Derivative prices are given by:

$$\Pi(t) = e^{-r\cdot(T-t)} E^{Q}_{r,t}\left[\Pi(T) \mid \mathcal{F}_t\right]$$

2.) The dynamics of Q is given by:

$$d\Pi_t = r\Pi_t dt + \sigma_{\Pi} \Pi_t d\nu_t$$

where ν_t is Q-martingale.

3.) $\dfrac{\Pi(t)}{B(t)}$ is martingale.

Siegel's Exchange Rate Paradox

We study again a market with two currencies, a domestic rate, r_d and a foreign rate, r_f. The exchange rate is given by $X(t)$. The process for the exchange rate is given by:

$$dX(t) = X(t)\left(r_d(t) - r_f(t)\right)dt + \sigma(t)\rho(t)X(t)dW^d(t)$$

In the formula above the mean rate of change for the exchange rate $X(t)$ is $r_d(t)$ - $r_f(t)$ under the domestic risk-neutral measure.

From the foreign perspective, the exchange rate is $1/X(t)$, and one should expect the mean rate of change of $1/X(t)$ to be $r_f(t)$ - $r_d(t)$.

This turns out not to be as straight forward as one might expect because of the convexity of the function $f(x) = 1/x$.

Siegel's Exchange Rate Paradox

Example:

Exchange rate of 0.9 euros to the dollar:

1 euro → 1.1111 dollars

If the dollar price of euro falls by 5%:

1 euro → 0.95 × 1.1111 = 1.0556 dollars

This is an exchange rate of 1/1.0556=0.9474 euros to dollars. The change from 0.9 euros to the dollar to 0.9474 euros to the dollar is a 5.26% (= 1/0.95 - 1) increase in the euro price of the dollar, not a 5% increase.

Siegel's Exchange Rate Paradox

If we take $f(x) = 1/x$ so that $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$, We obtain

$$d\left(\frac{1}{X}\right) = df\left(X\right) = f'\left(X\right)dX + \frac{1}{2}f''\left(X\right)\left(dX\right)^2$$

$$= \frac{1}{X}\left[\left(r_f - r_d\right)dt - \sigma dW^d\right] + \frac{1}{X}\sigma^2\left(dW^d\right)^2$$

$$= \frac{1}{X\left(t\right)}\left[\left(r_f - r_d + \sigma^2\right)dt - \sigma dW^d\right]$$

The mean rate of change under the domestic risk-neutral measure is $r_f - r_d + \sigma^2$ not $r_f - r_d$. We also observe that the correlation $\rho(t) = -1$. However, the asymmetry introduce by the convexity of $f(x) = 1/x$ is resolved if we switch to the foreign risk-neutral measure, which is the appropriate one for derivative security-pricing in the foreign currency.

Siegel's Exchange Rate Paradox

First, recall the relationship

$$dW^f(t) = -\sigma(t)dt + dW^d(t) \Rightarrow dW^d(t) = \sigma(t)dt + dW^f(t)$$

In term of $W^f(t)$, we may write

$$d\left(\frac{1}{X}\right) = \frac{1}{X}\left[\left(r_f - r_d\right)dt - \sigma dW^f\right]$$

Under the foreign risk-neutral measure, the mean rate of change for $1/X$ is $r_f - r_d$, as expect. Under the actual probability measure P, however, the asymmetry remains. By studying

$$dX(t) = \mu(t)X(t)dt + \sigma(t)X(t)dW(t)$$

We get

$$d\left(\frac{1}{X(t)}\right) = \left(\frac{1}{X(t)}\right)\left(-\mu(t) + \sigma^2(t)\right)dt - \frac{1}{X(t)}\sigma(t)dW(t)$$

Siegel's Exchange Rate Paradox

and we observe that both X and $1/X$ have the same volatility. But, their mean rates are not the negative of each other.

Delta-hedging

We will now study how to hedge using delta. Suppose that we take a short position in European call options on 100 000 stocks in ACME Inc. Suppose we have:

S	= 365,
X	= 370,
σ	= 20 %,
r	= 7 % and
$T - t$	= 0.25 year.

Via Black-Scholes formula, we get the option price 15.247 cash units so the total value is 1 524 738. As we will see, we are exposed for a risk.

Delta-hedging

First, suppose we have a naked position, e.g. we have no ownership in the underlying stock.

Study two cases:

1. At maturity, the stock price is < 370, (the option has no value) so we make a profit of 1 524 738.
2. At maturity, the stock price is 395 so we have to buy 100 000 stock for 395 cash units each and then sell them at a price of 370. The cost will be 100,000*(395 – 370) = 2.5 million, so we loose 975 262 cash units.

Delta-hedging

Next, suppose we have a covered position, i.e. we buy the stock at 365 each.

Study two cases:

1. At maturity, the stock price is 360, so we loose 500,000 selling the stocks, but get a total profit of 1 024 738. If the stock price goes below 350 we will make a negative profit totally.
2. At maturity, the stock price is 380. We sell the stocks at 370 each and make a total profit of 2 024 378.

Delta-hedging

We will now see how we can protect ourselves with a hedge. We will therefore calculate the number of stocks we have to buy, to hedge the options.
If $F(t, S)$ is the option value, N_c, the number of options and N_s the number of stocks, the total portfolio value is given by:

$$V = - N_c \cdot F(t, S) + N_s \cdot S$$

With a delta hedge:

$$\frac{dV}{dS} = 0 \quad \Rightarrow \quad N_s = N_c \cdot \frac{\partial F}{\partial S} = N_c \cdot \Delta = N_c \cdot N[d_1]$$

$$d_1 = \frac{1}{\sigma\sqrt{\Delta t}}\left\{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\Delta t\right\}$$

$$= \frac{1}{0.20\sqrt{0.25}}\left\{\ln\left(\frac{356}{370}\right) + \left(0.07 + \frac{1}{2}0.20^2\right)\cdot 0.25\right\} = 0.08894...$$

This gives $N[d_1] = 0.53550$, so we need $N_s = 53550$ stocks to hedge 100 000 options.

Naked

	1 short	2 -> 395
Call	TRUE	TRUE
S	365	395
K	370	370
T	0.25	
r	0.07	
vol	0.2	
Value	15.2474	-25.0000
N	100 000	100 000
Value	1 524 738	-2 500 000
		Resultat
		-975 262

Covered

	1 short	Long S	2 - > 360	2 - > 380
	TRUE			
	365	365	360	380
	370			370
	0.25			
	0.07			
	0.2			
	15.2474			
	100 000	100 000	100 000	100 000
	1 524 738	-36 500 000	36 000 000	37 000 000
			Resultat	Resultat
Total		-34 975 262	1 024 738	2 024 738

Delta-Gamma-hedging

If we also want to be Γ-neutral (i.e. have Γ = 0) we have to user a second option in our hedge:

$$\Gamma = \frac{N'[d_1]}{S\sigma\sqrt{\Delta t}}$$

Suppose there exist a put option *p*, at strike price of 355 cash units. Then our portfolio is

$$V = N_p \cdot F_p(t, S) - N_c \cdot F_c(t, S) + N_s \cdot S$$

and

$$\frac{\partial V}{\partial S} = \frac{\partial^2 V}{\partial S^2} = 0 \quad \Rightarrow \quad \{\Delta_p = \Delta_c - 1\} \quad \Rightarrow$$

$$\begin{cases} N_s = N_c \cdot \Delta_c - N_p \cdot \Delta_p \\ 0 = N_c \cdot \Gamma_c - N_p \cdot \Gamma_p \end{cases}$$

Delta-Gamma-hedging

If we also want to remove the sensitivity in Δ, we also must have Γ = *0*. Since Γ = *0* for stocks we have to use one more option. Given a portfolio Π with a stock *S* and two derivatives, *F* and *G*. We want to choose X_F and X_G so that the total portfolio becomes both *Δ*- and *Γ*-neutral:

$$\begin{cases} N_p = 100.000 \cdot \dfrac{\Gamma_c}{\Gamma_p} \\ N_s = 100.000 \cdot \Delta_c - 100.000 \Delta_p \cdot \dfrac{\Gamma_c}{\Gamma_p} \end{cases}$$

Giving:

$$\begin{cases} N_p = 160.800 \\ N_s = 102.300 \end{cases}$$

Delta-Gamma - Hedge

If the stock price of some share is 35 and we want to hedge 1000 stocks of this share at a price of 35. On the market there exist options with strikes 30 and 37. The risk free interest rate is estimated to 4.5 % and the time to maturity of the options is 102 days.
The volatility is estimated to 37.5%. If we use the formulas above on a call option with strike 30 and a put option with strike 37, we will find that if we buy 548 put options and go short in 847 call options we will hedge our 1000 stocks.

In the figure below we illustrate the total portfolio value when the stock price varies between 0 and 70. As we can see, the hedge is very good in a region between 28 and 42. We also observe that we earn 2000 in the hedge.

	Call	TRUE	FALSE	Stock	TRUE	FALSE	Stock
	S	35	35	35	35	35	35
	K	30	37		37	30	
102 days	T	0.2795	0.2795		0.2795	0.2795	
	r	0.045	0.045		0.045	0.045	
	vol	0.375	0.375		0.375	0.375	
	N	-847	548	1 000	-1386	2 142	1 000
	Value	6.0862	3.6589	35 000	2.1213	0.7113	35 000
	Delta	0.8264	-0.5469		0.4531	-0.1736	
	Gamma	0.0370	0.0571		0.0571	0.0370	
	Vega	4.7445	7.3303		7.3303	4.7445	
	Value	-5 156.02	2 006.30	35 000	-2 940.97	1 523.52	35 000
			Total	31 850		Total	33 583

TRUE	FALSE	Stock	TRUE	FALSE	Stock	TRUE	FALSE	Stock	TRUE	FALSE	Stock
30	30	30	30	30	30	40	40	40	40	40	40
30	37		37	30		30	37		37	30	
0.2795	0.2795		0.2795	0.2795		0.2795	0.2795		0.2795	0.2795	
0.045	0.045		0.045	0.045		0.045	0.045		0.045	0.045	
0.375	0.375		0.375	0.375		0.375	0.375		0.375	0.375	
-847	548	1 000	-1386	2 142	1 000	-847	548	1 000	-1386	2 142	1 000
2.5460	7.0885	30 000	0.5509	2.1711	30 000	10.5676	1.5988	40 000	5.0612	0.1927	40 000
0.5646	-0.8147		0.1853	-0.4354		0.9467	-0.2892		0.7108	-0.0533	
0.0662	0.0449		0.0449	0.0662		0.0137	0.0431		0.0431	0.0137	
6.2438	4.2374		4.2374	6.2438		2.2942	7.2283		7.2283	2.2942	
-2 156.94	3 886.86	30 000	-763.72	4 650.58	30 000	-8 952.59	876.70	40 000	-7 016.93	412.82	40 000
	Total	31 730		Total	33 887		Total	31 924		Total	33 396

Delta-Gamma - Hedge