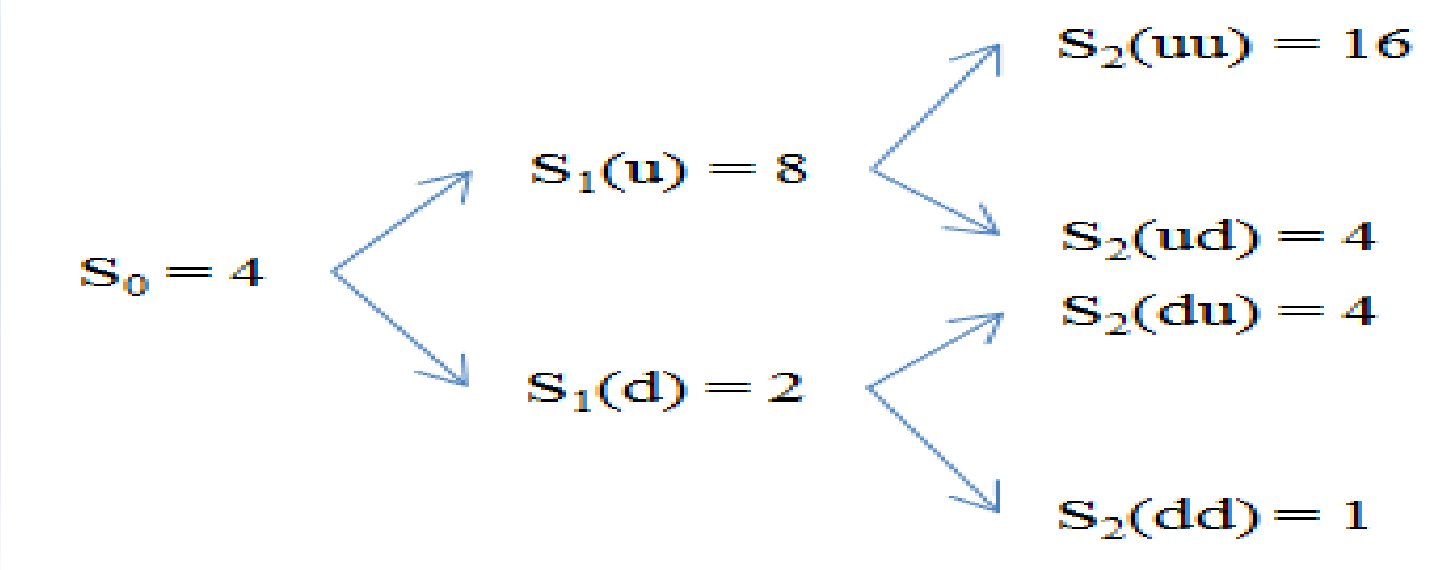
Stopping times and American options

For an American contract with a value process $V_n = g(S_n)$ we define the backward recursion as

$$\begin{cases} V_n(x) = g(x) \\ V_k(x) = \max\left\{ \frac{1}{1+r}\left(\tilde{p} V_{k+1}(ux) + \tilde{q} V_{k+1}(dx) \right), g(x) \right\} \end{cases}$$

Let us study a two-period binomial tree for an American put option with $S_0 = 4$, $u = 2$, $d = ½$, $p = q = ½$ and $r = ¼$ with a strike $K = 5$.



Stopping times and American options.

At maturity we have the value: $V_2 = (5 - S_k)^+$

We have: $V_{uu} = 0$, $V_{ud} = V_{du} = 1$ and $V_{dd} = 4$. The tree gives us:

$$V_u = \max\left\{\frac{1}{1+r}\left[\tilde{p}V_{uu} + \tilde{q}V_{ud}\right], (5-8)^+\right\} = \max\left\{\frac{4}{5}\left[\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1\right], 0\right\} = 0.40$$

$$V_d = \max\left\{\frac{1}{1+r}\left[\tilde{p}V_{ud} + \tilde{q}V_{dd}\right], (5-2)^+\right\} = \max\left\{\frac{4}{5}\left[\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4\right], 3\right\} = 3$$

$$V = \max\left\{\frac{1}{1+r}\left[\tilde{p}V_u + \tilde{q}V_d\right], (5-4)^+\right\} = \max\left\{\frac{4}{5}\left[\frac{1}{2}\cdot 0,4 + \frac{1}{2}\cdot 3\right], 1\right\} = 1,36$$

$$\Delta_{k-1} = \frac{V_k(u) - V_k(d)}{S_k(u) - S_k(d)}$$

we find $\Delta_0 = (0.4 - 3.0)/(8.0 - 2.0) = -0.4333$

Stopping times and American options

For k = 1 the values:

$$4 = V_{dd} = S_2(dd)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d)) \quad \Rightarrow \quad \Delta_1(d) = -1.83$$
$$1 = V_{du} = S_2(du)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d)) \quad \Rightarrow \quad \Delta_1(d) = -0.16$$

($X_1(d) = V_d$). We see that:

$$1 = V_{du} = S_2(du)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d)) = 4 \cdot \Delta_1 + \frac{5}{4}(3 - 2 \cdot \Delta_1)$$
$$=> \Delta_1 (4 - 2.5) = 1 - 3.75 = -2.75$$
$$=> \Delta_1 = -1.83$$

$$4 = V_{dd} = S_2(dd)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d)) = 1 \cdot \Delta_1 + \frac{5}{4}(3 - 2 \cdot \Delta_1)$$
$$=> \Delta_1 (1 - 2.5) = 4 - 3.75 = 0.25$$
$$=> \Delta_1 = -0.16$$

Stopping times and American options

If this was a European option $X_1(d) = S_1(d) = 2$ and Δ_1 should be equal (= -1) :

$$V_u = \frac{1}{1+r}[\tilde{p}V_{uu} + \tilde{q}V_{ud}] = \frac{4}{5}\left[\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1\right] = 0.40$$

$$V_d = \frac{1}{1+r}[\tilde{p}V_{ud} + \tilde{q}V_{dd}] = \frac{4}{5}\left[\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4\right] = 2$$

$$V = \frac{1}{1+r}[\tilde{p}V_u + \tilde{q}V_d] = \frac{4}{5}\left[\frac{1}{2}\cdot 0,4 + \frac{1}{2}\cdot 2\right] = 0,96$$

so

$$1 = V_{du} = S_2(du)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d)) = 4\cdot\Delta_1 + \frac{5}{4}(2 - 2\cdot\Delta_1)$$

$$=> \Delta_1(4 - 2.5) = 1 - 2.5 = -1.5$$

$$=> \Delta_1 = -1.0$$

$$4 = V_{dd} = S_2(dd)\Delta_1(d) + (1+r)(X_1(d) - \Delta_1(d)S_1(d)) = 1\cdot\Delta_1 + \frac{5}{4}(2 - 2\cdot\Delta_1)$$

$$=> \Delta_1(1 - 2.5) = 4 - 2.5 = -1.5$$

$$=> \Delta_1 = -1.0$$

Stopping times and American options

The value of a hedged portfolio with an American option is given by:

$$X_{k+1} = S_{k+1}\Delta_k + (1+r)(X_k - \Delta_k S_k - C_k)$$
$$= (1+r)X_k + \Delta_k\left(S_{k+1} - (1+r)S_k - (1+r)C_k\right)$$

where C_k is the part consumed at time $t = k$.

Properties:

- The discounted portfolio value is a super martingale.
- The value satisfy $X_k \geq g(S_k)$, $k = 0, 1, ..., n$.
- The value process is the process with the lowest value with these properties.

Stopping times and American options

Question: When do we consume?
Answer: If:

$$E\left[(1+r)^{-(k+1)}V_{k+1}(S_{k+1}) \mid \mathcal{F}_k\right] < (1+r)^{-k}V_k(S_k) \quad \Rightarrow$$

$$\frac{1}{1+r}E\left[V_{k+1}(S_{k+1}) \mid \mathcal{F}_k\right] < V_k(S_k)$$

If the holder of the option doesn't exercise, then we can consume and close the gap! In that case, when $X_k = V_k(S_k)$ for all values of k and where

$$\begin{cases} V_n(x) = g(x) \\ V_k(x) = \max\left\{\frac{1}{1+r}\left(\tilde{p}V_{k+1}(ux) + \tilde{q}V_{k+1}(dx)\right), g(x)\right\} \end{cases}$$

Stopping times and American options

In the previous example e.g., $V_1(S_1(u)) = 3$, $V_2(S_2(ud)) = 1$, $V_2(S_2(uu)) = 4$,we get

$$\frac{1}{1+r} E\left[V_2(S_2) \mid \mathcal{F}_1\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4\right] = 2$$

If the holder don't exercise at $t = 1$ we can consume one cash unit and hedge as

$$\Delta_k = \frac{V_{k+1}(uS_k) - V_{k+1}(dS_k)}{(u-d)S_k}$$

As we can see, from the holder’s point of view, it is optimal to exercise when $V_k(S_k) = g(S_k)$.

Stopping times

Definition: Given the probability space $(\Omega, \mathcal{F}, P)$ and the filtration $\{\mathcal{F}_k\}_{k=0}^{n}$ of $\mathcal{F}$ we define the **stopping time** as a stochastic variable $\tau : \Omega \rightarrow \{0, 1, ..., n\} \cup \{\infty\}$ such as

$$\{\omega \in \Omega;\ \tau(\omega) = k\} \in \mathcal{F}_k \ \forall\ k = 0, 1, ..., n, \infty$$

Example – Stopping time

We define (from the tree above)

$$\tau(\omega) = \min\{k \mid V_k(S_k) = (5 - S_k)^+\}$$

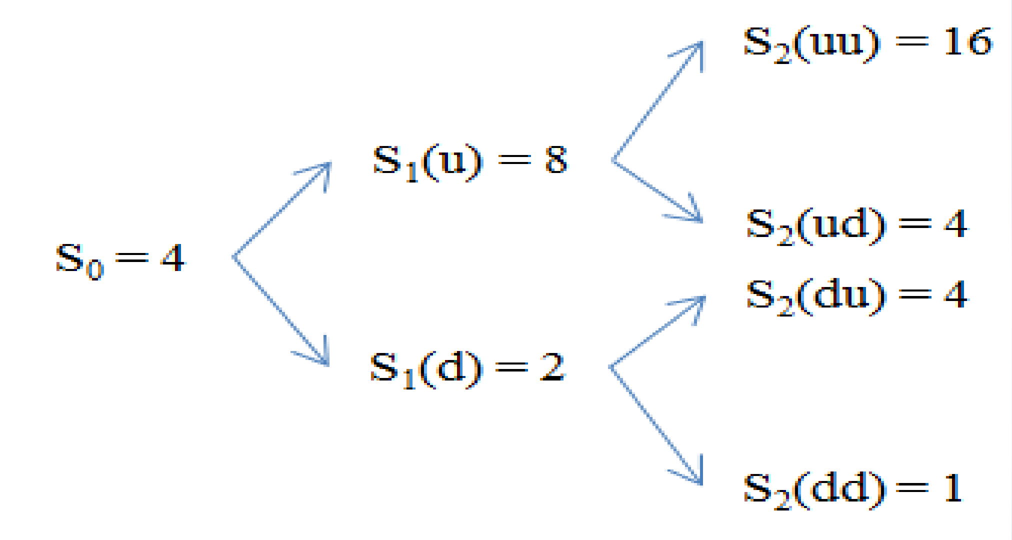
This stopping time is the time when the option value for the first time is equal to the instantaneous value. This time is the optimal time to exercise the option. A stopping time is characterized by the fact that we at every time $t < \tau$ can decide if τ has occurred or not, based on the information we really have at time t. Remark:

$$\tau(\omega) = \begin{cases} 1 & \text{if } \omega = A_d \\ 2 & \text{if } \omega = A_u \end{cases}$$

$$\{\omega : \tau(\omega) = 0\} = \varnothing \in \mathcal{F}_0$$
$$\{\omega : \tau(\omega) = 1\} = A_d \in \mathcal{F}_1$$
$$\{\omega : \tau(\omega) = 2\} = A_u \in \mathcal{F}_2$$

Markov processes

We start by studying a European lookback option with values S_0 = 4, u = 2, d = ½, p = q = ½ and r = ¼ with a strike price K = 5 with a two-period binomial model:



The value of the lookback option is given by:

$$V_2 = \max_{0 \le t \le 2} (S_k - 5, 0)$$

Markov processes

We study the evolution backwards to calculate the value, thereby the name lookback. We have: $V_{uu} = 11$, $V_{ud} = 3$, $V_{du} = 0$ and $V_{dd} = 0$. (Remark $V_{ud} \neq V_{du}$). By travelling backwards in the tree we get:

$$V_u = \frac{1}{1+r}\left[pV_{uu} + qV_{ud}\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 11 + \frac{1}{2}\cdot 3\right] = 5.60$$

$$V_d = 0$$

$$V = \frac{4}{5}\cdot\frac{1}{2}\cdot 5.60 = 2.24$$

with

$$\Delta_{t-1} = \frac{V_t(u) - V_t(d)}{S_t(u) - S_t(d)}$$

we get $\Delta_0 = (5.6 - 0.0)/(8 - 2) = 0.93$, $\Delta_1(u) = (11.0 - 3.0)/(16 - 4) = 0.67$ and $\Delta_1(d) = 0$. If we now sell one option at $X_0 = 2.24$ and hedge us with Δ_0 shares we get:

Markov processes

$$X_1(u) = \Delta_0 S_1(u) + (1+r)(X_0 - \Delta_0 S_0)$$
$$= 0.93 \cdot 8 + (1+0.25)(2.24 - 0.93 \cdot 4)$$
$$= 5.60$$

$$X_1(d) = \Delta_0 S_1(d) + (1+r)(X_0 - \Delta_0 S_0)$$
$$= 0.93 \cdot 2 + (1+0.25)(2.24 - 0.93 \cdot 4)$$
$$= 0$$

$$X_2(uu) = \Delta_1(u) S_2(uu) + (1+r)(X_1(u) - \Delta_1(u) S_1(u))$$
$$= 0.67 \cdot 16 + (1+0.25)(5.60 - 0.67 \cdot 8)$$
$$= 11.0$$

$$X_2(ud) = \Delta_1(u) S_2(ud) + (1+r)(X_1(u) - \Delta_1(u) S_1(u))$$
$$= 0.67 \cdot 4 + (1+0.25)(5.60 - 0.67 \cdot 8)$$
$$= 3.0$$

Markov processes

An ordinary European call option with the same data as above $V_2 = (S_k - 5)^+$ gives: V_{uu} = 11, V_{ud} = V_{du} = 0 and V_{dd} = 0. (Remark! V_{ud} = V_{du}). Further:

$$V_u = \frac{1}{1+r}\left[pV_{uu} + qV_{ud}\right] = \frac{4}{5}\left[\frac{1}{2}\cdot 11 + \frac{1}{2}\cdot 0\right] = 4.40$$

$$V_d = 0$$

$$V = \frac{4}{5}\cdot\frac{1}{2}\cdot 4.40 = 1.76$$

with

$$\Delta_{t-1} = \frac{V_t(u) - V_t(d)}{S_t(u) - S_t(d)}$$

we get Δ_0 = (4.4 – 0.0)/(8 – 2) = 0.733, $\Delta_1(u)$ = (11.0 – 0.0)/ (16 – 4) = 0.917 and $\Delta_1(d)$ = 0. If we now sell one option at X_0 = 1.76 and hedge us with Δ_0 shares we get:

Markov processes

$$X_1(u) = \Delta_0 S_1(u) + (1+r)(X_0 - \Delta_0 S_0)$$
$$= 0.733 \cdot 8 + (1+0.25)(1.76 - 0.733 \cdot 4)$$
$$= 4.40$$

$$X_1(d) = \Delta_0 S_1(d) + (1+r)(X_0 - \Delta_0 S_0)$$
$$= 0.743 \cdot 2 + (1+0.25)(1.76 - 0.733 \cdot 4)$$
$$= 0$$

A general problem:

For a model with n periods, we have Ω 2^n elements giving 2^n equations. For a three months option we have 66 trading days and with a period length of one day we get $2^{66} \approx 7*10^{19}$ equations.

Markov processes

Solution:

We can solve this in three ways:

1. By simulations and averaging.
2. Approximate in continuous time. This gives a PDE-theory.
3. Using a Markov structure.

What we are doing in the binomial model is exactly 3.) above. Instead or four values at $n = 2$ (V_{uu}, V_{ud}, V_{du} and V_{dd}) we have three, because of $V_{ud} = V_{du}$. This gives us $n + 1$ equations instead of 2^n!

Markov processes

Definition: Let (Ω, $\mathcal{F}$, P) be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$
An adapted process is said to be a **Markov process** with respect to the filtration $(\mathcal{F}_t)$ if

$$E[f(X_t) \mid \mathcal{F}_s] = E[f(X_t) \mid X_s]$$ almost sure for all $t \geq s \geq 0$.

for every bounded real-valued Borel function f defined on $\mathbb{R}^d$.

In words: If we are studying a path, described by a Geometrical Brownian Motion (GBM) from 0 till t_0 and want to estimate the value of $f(X(t_1))$, the only relevant information is the value of $X(t_0)$.

Example: The stock price in the binomial model is a Markov process.

Radon-Nikodym

Theorem: Let P and Q being two probability measures on $(\Omega, \mathcal{F})$. Suppose that for each $A \in \mathcal{F}$ with $P(A) = 0$, and also $Q(A) = 0$, then we say that Q is **absolute continuous** with respect to P. Furthermore, then there exists a stochastic variable Z (≥ 0) such as:

$$Q(A) = \int_{\Omega} Z dP(A)$$

We name Z as the Radon-Nikodym derivative of Q with respect to P.

It follows trivially from the definition of the derivative that, when P and Q are probability measures over the probability space Ω and X is a random variable. Then

$$E^{Q}[X] = \int_{\Omega} X dQ = \int_{\Omega} X \frac{dQ}{dP} dP = E^{P}\left[\frac{dQ}{dP} X\right]$$

Radon-Nikodym

If P at the same time is absolute continuous with respect to Q we say that P and Q are equivalent. I.e., if and only if $Q(A) = 0$ exactly when $P(A) = 0$ we have:

$$E^Q[X] = E^P[XZ] \quad \forall X$$

$$E^P[Y] = E^Q\left[Y\frac{1}{Z}\right] \quad \forall Y$$

Example: Let $\Omega = \{uu, ud, du, dd\}$, $P(u) = 1/3$, $P(d) = 2/3$ and $Q(u) = Q(d) = 1/2$. Define $Z(\omega)$ as $Q(\omega)/P(\omega)$ Then:

$Z(uu) = (1/2)^2/(1/3)^2 = 9/4$, $Z(ud) = 9/8$, $Z(du) = 9/8$ and $Z(dd) = 9/16$

Radon-Nikodym

In the financial analysis the Radon-Nikodym derivative is used to change measures. If we have a sample space Ω, with market probabilities P and if Q is the risk-neutral probability distribution, then we can find a transformation between P and Q with the help of the Radon-Nikodym derivative. If $P(\omega) > 0$ and $Q(\omega) > 0$ for all $\omega \in \Omega$, P and Q are equivalent. We write this as $Q \sim P$. If P and Q are absolute continuous we write this as $Q << P$.

Two measures are equivalent if they have the same sample space and the same set of "possibilities". Note the use of the word possibilities instead of probabilities. The two measures can have different probabilities for each outcome but must agree on what is possible.

Another way to formulate the Radon-Nikodym is using two different measures, μ and ν on (Ω, X). Absolute continuity $\Leftrightarrow \mu << \nu$ and equivalence $\mu \sim \nu$. If $\nu \sim \mu$, i.e. they have exactly the same empty measure Ø, then we write the Radon-Nikodym derivative as:

$$f = \frac{d\nu}{d\mu} \quad \Leftrightarrow \quad d\nu(x) = f(x) \cdot d\mu(x)$$

Radon-Nikodym

With this definition we can always find *f*, also on point sets:

$$f(n) = \begin{cases} \nu(n)/\mu(n) & \text{if } \mu(n) \neq 0 \\ 0 & \text{else} \end{cases}$$

Remark, if we make a σ-algebra finer and finer we may lose the absolute continuity. Suppose we have a given probability spaces $(\Omega, \mathcal{F}, P)$, with a filtration $\underline{\mathcal{F}}$ on the interval $[0, T]$. Then, if $L_T \geq 0$ is a $\mathcal{F}$-measurable stochastic variable we can find a new measure Q on $(\Omega, \mathcal{F}_T)$ via: $dQ = L_T dP$ where Q will be a probability measure if $E^P[L_T] = 1$:

$$\int_\Omega dQ = \int_\Omega L_T dP = E^P\left[L_T\right] = 1$$

Radon-Nikodym

Definition: Z_k says to be a P-martingale if

$$Z_k = E^P\left[Z \mid \mathcal{F}_k\right] \quad k = 0,1,\ldots,n$$

$$E^P\left[Z_{k+1} \mid \mathcal{F}_k\right] = E^P\left[E^P\left[Z \mid \mathcal{F}_{k+1}\right] \mid \mathcal{F}_k\right] = E^P\left[Z \mid \mathcal{F}_k\right] = Z_k$$

Lemma: If X is $\mathcal{F}_k$-measurable and $0 \le j \le k$, then

$$E^Q\left[X \mid \mathcal{F}_j\right] = \frac{1}{Z_j} E^P\left[X Z_k \mid \mathcal{F}_j\right]$$

Theorem: L (as above) is a $(\mathcal{F}, P)$-martingale.

Proof: See Lecture Notes

Brownian motion

When studying lattices-models of random processes one naturally wonders if model asset prices make sense in the limit that the step size goes to zero? To answer this question we now advance to continuous random processes.

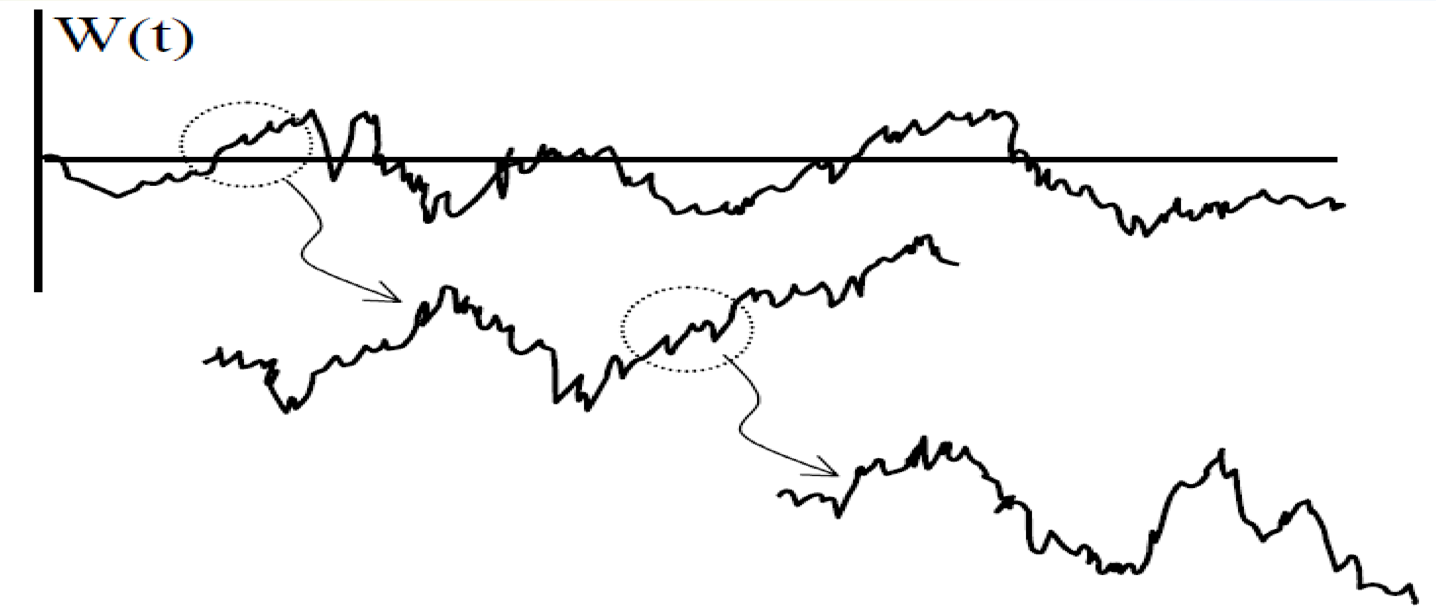
We are aiming to develop models based on stochastic differential equations:

$$dS = a(t,S)dt + b(t,S)dW$$

for the asset price $S(t)$, where the $a(t, S)dt$ term accounts for "deterministic motions", and the other term $b(t, S)dW$ accounts for "random motions".

The first step in developing such models is to decide what we should use for $W(t)$, the random part of the model. No matter how we sub-divide it, the curve $W(t)$ should still be random and composed of pieces with identical statistical properties, because asset prices appear random on even very fine time scales

Brownian motion



First, $W(t)$ must have independent increments. For any date τ and for any $\Delta\tau > 0$, $\Delta W = W(\tau + \Delta\tau) - W(\tau)$ is independent of $W(t)$ for all $t \leq \tau$. So increments are independent of everything on or before τ.

In particular, $W(t_2) - W(t_1)$ and $W(t_3) - W(t_4)$ are independent whenever $t_1 \leq t \leq t_2$ and $t_3 \leq \mathrm{t} \leq t_4$ don't overlap. I.e., $W(t) - W(\tau)$ for $t > \tau$ does not depend on how one got to $W(\tau)$.

As we shall see, this is a very powerful simplifying assumption.

Brownian motion

Second, increments $\Delta W \equiv W(t_2) - W(t_1)$ are Gaussian random variables with mean 0 and variance $\Delta t \equiv t_2 - t_1$. That is,

$$\Delta W \equiv W(t_2) - W(t_1) = \sqrt{t_2 - t_1}\,\xi$$

where ξ is $N(0, 1)$, i.e., ξ is a Gaussian random variable with mean zero and variance 1. The reason we want ΔW to have mean zero is because we want it to represent the random part; any non-zero mean would represent a deterministic piece which we could put in the drift term.

The fact that ΔW is Gaussian with variance Δt follows directly from our desire to have $W(t)$ to be subdivideable into finer and finer intervals, each with identical statistical properties.

See further details in Lecture Notes.

Brownian motion

$$\begin{cases} dS(t) = \alpha \cdot S(t) \cdot dt + \sigma \cdot S(t) \cdot dW \\ S(0) = s \end{cases}$$

$$S(t) = s \cdot e^{\left\{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma \cdot W(t)\right\}}$$

Since $W(t) - W(t_0)$ is normal distributed with mean zero and variance $(t - t_0)$: $N[0, (t - t_0)]$ we know that Z must be normal distributed as $(\mu = \alpha)$:

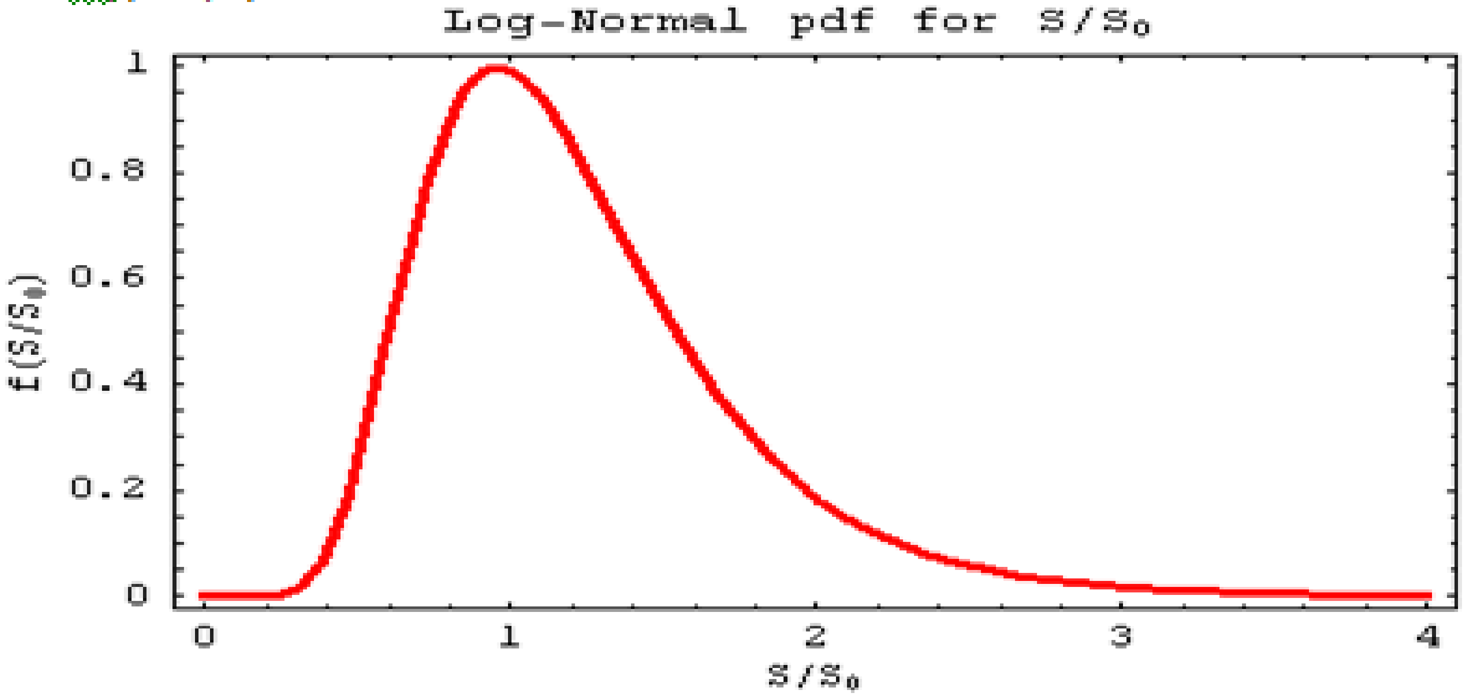
$$Z \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)(t - t_0), \sigma^2 (t - t_0)\right]$$

Therefore $S(t)/S(t_0)$ follows a log-normal distribution:

$$g(S(t)) = \frac{1}{\sigma S(t)\sqrt{2\pi(t - t_0)}} \exp\left\{-\frac{\left(\ln\{S(t)/S(t_0)\} - (\mu - \sigma^2/2)(t - t_0)\right)^2}{2\sigma^2 (t - t_0)}\right\}$$

Brownian motion

Example: With $\sigma^2 = 0.4$, $\mu = 0.16$ and $(t - t_0) = 1$ we get the following probability distribution for $S(t)/S(t_0)$.



Stochastic integration

To understand stochastic integration we will start by studying the integral

$\int g(s)dW(s)$. We will do this in a few simple steps:

1.) Split the interval $[0, t]$ into equal parts $0 = t_0 < t_1 \ldots < t_n = t$.
2.) For each outcome ω define an integral:

$$I_n(\omega) = \sum g(\xi_k, \omega)\left[W(t_{k+1}, \omega) - W(t_k, \omega)\right]$$

3.) Let $n \to \infty$ and hope for $I_n \to I$

Let $g = W$ and study the integral $\int W(s)dW(s)$ by defining A_n and B_n:

Stochastic integration

$$\begin{cases} A_n = \sum_{k=1}^{n} W(t_k)[W(t_{k+1}) - W(t_k)] & \xi_k = t_k \\ B_n = \sum_{k=1}^{n} W(t_{k+1})[W(t_{k+1}) - W(t_k)] & \xi_k = t_{k+1} \end{cases}$$

$$\begin{aligned} A_n &= \lim_{n\to\infty} \sum_{k=1}^{n} W(t_k)[W(t_{k+1}) - W(t_k)] \\ &= \frac{1}{2}\lim_{n\to\infty} \sum_{k=1}^{n} \left[W(t_{k+1}) + W(t_k) - \left(W(t_{k+1}) - W(t_k)\right)\right]\left(W(t_{k+1}) - W(t_k)\right) \\ &= \frac{1}{2}\lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(W(t_{k+1}) + W(t_k)\right)\left(W(t_{k+1}) - W(t_k)\right)\right] - \frac{1}{2}\lim_{n\to\infty} \sum_{k=1}^{n} \left(W(t_{k+1}) - W(t_k)\right)^2 \\ &= \frac{1}{2}\lim_{n\to\infty} \sum_{k=1}^{n} \left[W^2(t_{k+1}) - W^2(t_k)\right] - \frac{1}{2}\lim_{n\to\infty} \sum_{k=1}^{n} \left(W(t_{k+1}) - W(t_k)\right)^2 \\ &= \frac{1}{2} W^2(t) - \frac{1}{2} t \end{aligned}$$

Stochastic integration

$$
\begin{aligned}
B_n &= \lim_{n\to\infty}\sum_{k=1}^{n} W(t_{k+1})\left[W(t_{k+1}) - W(t_k)\right] \\
&= \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left[W(t_{k+1}) + W(t_k) + \left(W(t_{k+1}) - W(t_k)\right)\right]\left(W(t_{k+1}) - W(t_k)\right) \\
&= \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left[\left(W(t_{k+1}) + W(t_k)\right)\left(W(t_{k+1}) - W(t_k)\right)\right] + \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left(W(t_{k+1}) - W(t_k)\right)^2 \\
&= \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left[W^2(t_{k+1}) - W^2(t_k)\right] + \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left(W(t_{k+1}) - W(t_k)\right)^2 \\
&= \frac{1}{2}W^2(t) + \frac{1}{2}t
\end{aligned}
$$

$$
\begin{cases}
A_n + B_n = W^2(t) \\
B_n - A_n = \sum_{k=0}^{n-1}(\Delta W_k)^2 = S_n
\end{cases}
\qquad
\lim_{n\to\infty} S_n(t) = t
$$

Stochastic integration

By letting $B_n - A_n \to t \;\Rightarrow A_n \to A$ and $B_n \to B$ where

$$\begin{cases} A = \dfrac{W^2(t)}{2} - \dfrac{t}{2} \\ B = \dfrac{W^2(t)}{2} + \dfrac{t}{2} \end{cases}$$

We then observe that ξ_k effects the integral concept and

$$\begin{cases} \int_0^t W(s)dW(s) = \dfrac{W^2(t)}{2} - \dfrac{t}{2} & \text{This is called (the forward- or) the Itô-integral and} \\ \int_0^t W(s)dW(s) = \dfrac{W^2(t)}{2} + \dfrac{t}{2} & \text{this the backward integral.} \end{cases}$$

Stochastic integration

We will use the Itô stochastic integral for an important reason:
In all natural cases unknown future events cannot affect the present.
This means that the values of a function $G(t)$ is non-anticipating in that it cannot be used to predict the future increment in dX.
This is of course equivalent in saying $G(t)$ is a martingale since what we mean is exactly that

$$E_s\left[G(t)\right] = G(s) \quad \text{för} \quad s \leq t.$$

We only know what is the present value of $G(t)$, which corresponds to that at the beginning integrating interval. For this reason it is more appropriate to use the Itô integral.

Classifications of PDE's

A general quadratic surface can be described by the expression

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Depending on the values of the constants (*A*, *B*, *C*, *D*, *E* and *F*), different geometrical objects will be represented:

$A = C, B = 0$: $\Rightarrow$ a Circle,

$B^2 - 4AC < 0$: $\Rightarrow$ an Ellipse,

$B^2 - 4AC = 0$: $\Rightarrow$ a Parabola and

$B^2 - 4AC > 0$: $\Rightarrow$ a Hyperbola

Similarly, we classify second order partial differential equations by the expression:

$$A\frac{\partial^2 F(x,y)}{\partial x^2} + B\frac{\partial^2 F(x,y)}{\partial x \partial y} + C\frac{\partial^2 F(x,y)}{\partial y^2} + D\frac{\partial F(x,y)}{\partial x} + E\frac{\partial F(x,y)}{\partial y} + F(x,y) = 0$$