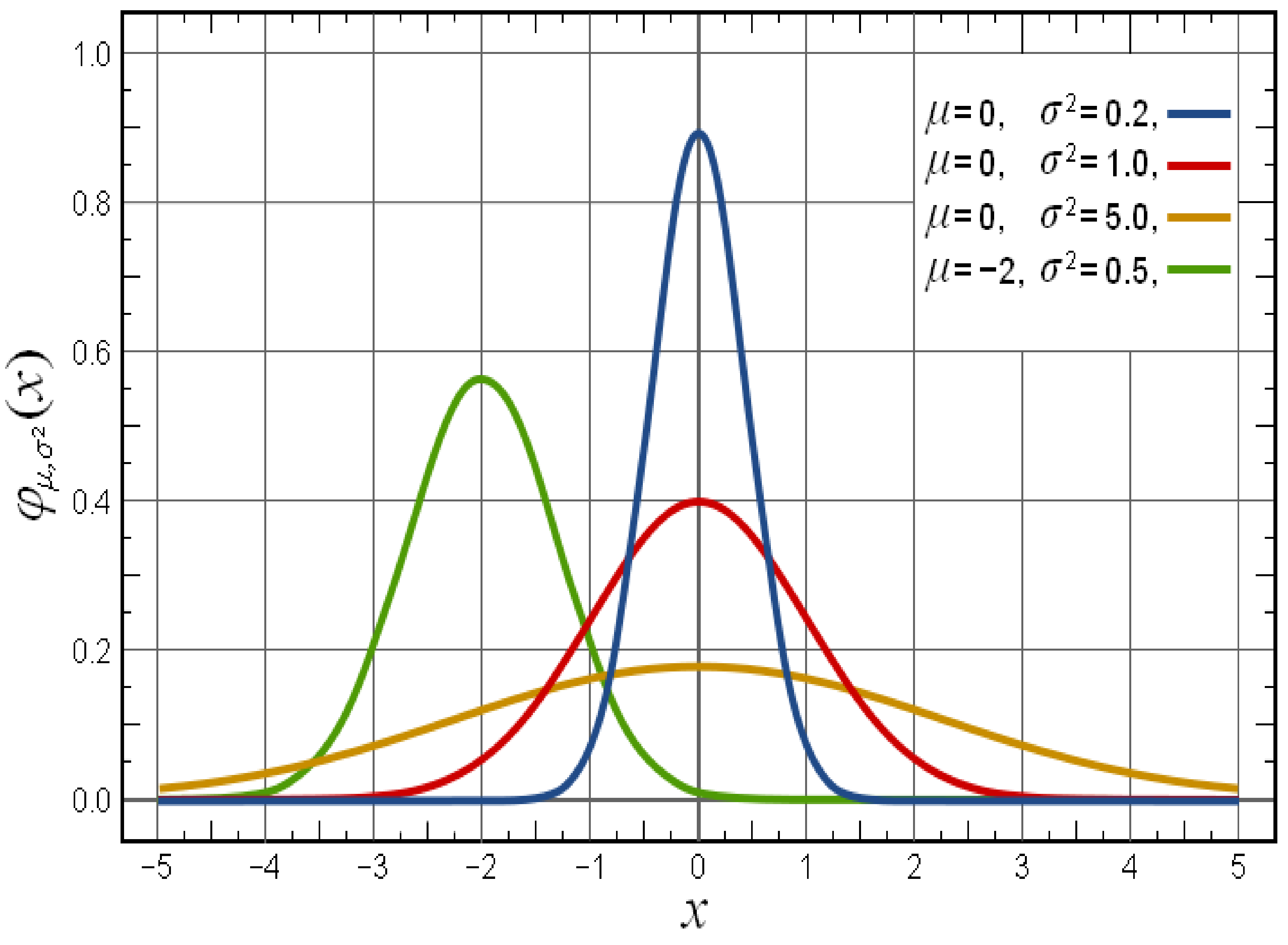
Properties of normal and log-normal distributions

If the density function φ is given by:

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\} = N'\left(\frac{x-m}{\sigma}\right)$$

we say that X has a Gaussian (or normal) distribution, with mean m and variance σ^2. In this case we say that X is an $N(m, \sigma^2)$ random variable.



1.0
0.8
0.6
0.4
0.2
0.0
$\varphi_{\mu,\sigma^2}(x)$
−5
−4
−3
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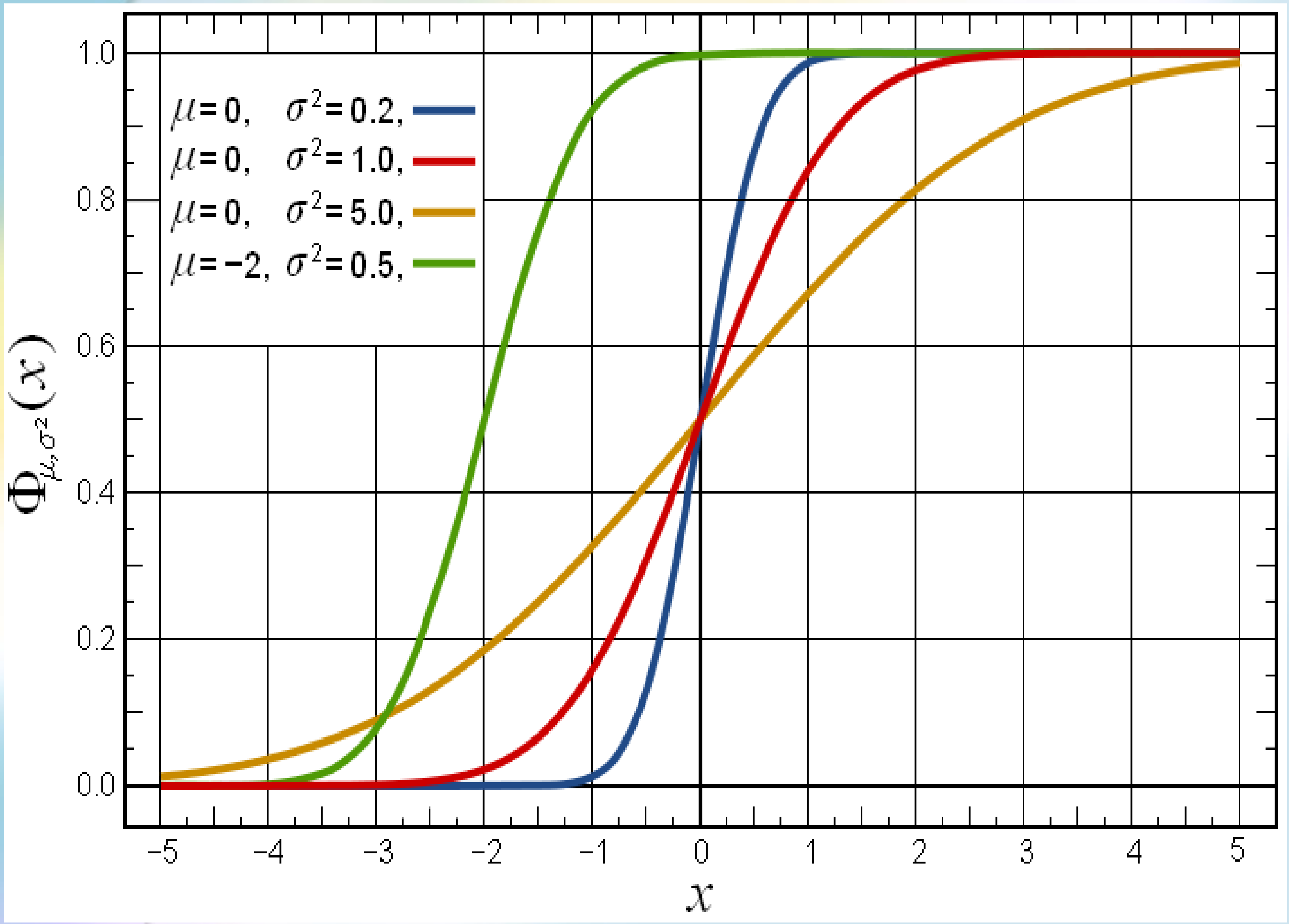
−1
0
1
2
3
4
5
x
$\mu=0, \quad \sigma^2=0.2,$
$\mu=0, \quad \sigma^2=1.0,$
$\mu=0, \quad \sigma^2=5.0,$
$\mu=-2, \ \sigma^2=0.5,$



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$\mu=-2, \sigma^2=0.5,$
1.0
0.8
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0.4
0.2
0.0
$\Phi_{\mu,\sigma^2}(x)$
−5
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−3
−2
−1
0
1
2
3
4
5
x

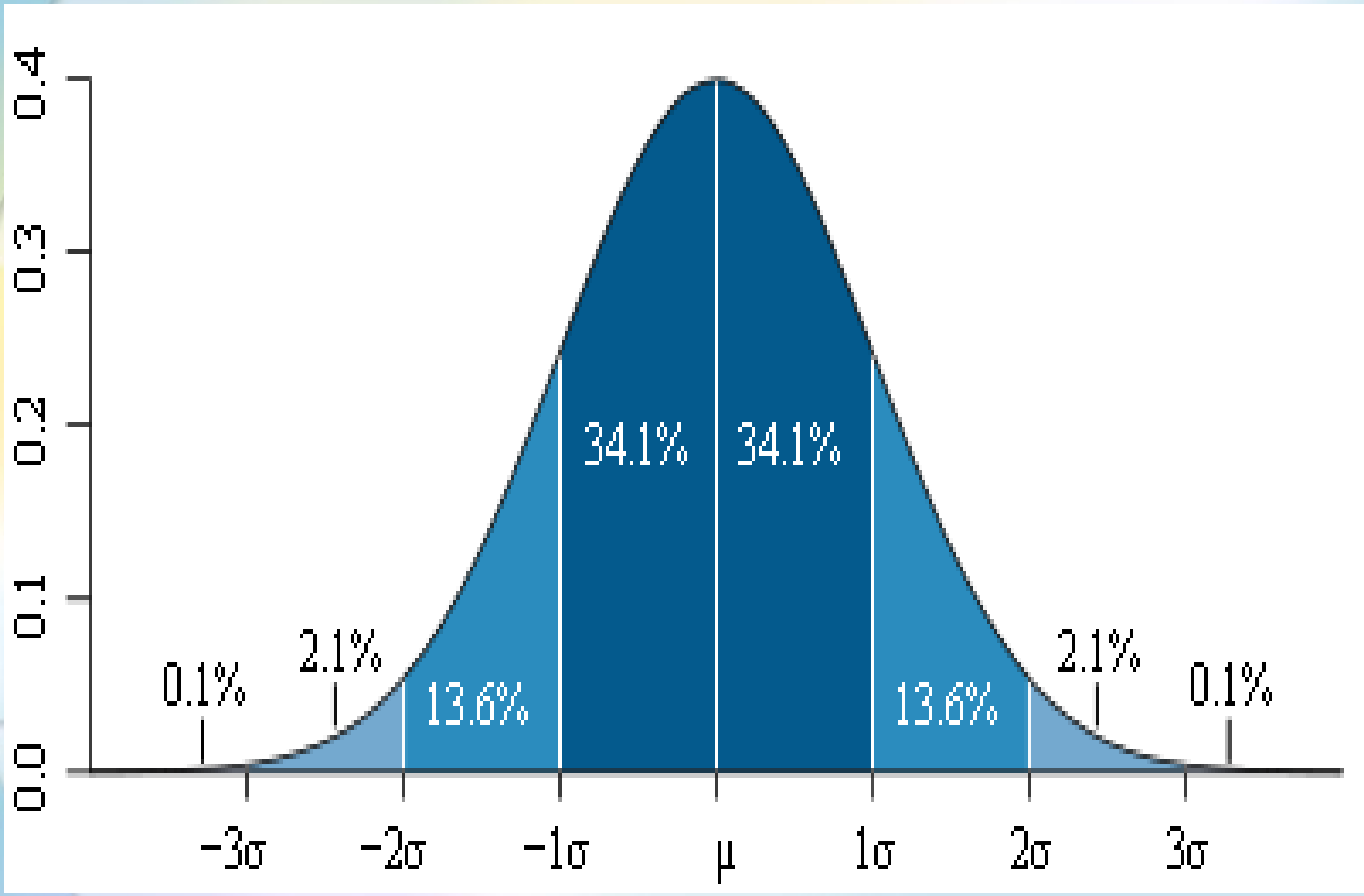
$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$$

$$Q(x) = \int_{x}^{\infty} \varphi(t)\,dt = 1 - \Phi(x)$$

The family of normal distributions is closed under linear transformations. That is, if X is normally distributed with mean μ and variance σ^2, then a linear transform $aX + b$ (for some real numbers $a \neq 0$ and b) is also normally distributed:

$$aX + b \sim \mathcal{N}(a\mu + b,\, a^2\sigma^2)$$

$$\Pr(X \leq x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$


0.4
0.3
0.2
0.1
0.0
0.1%
2.1%
13.6%
34.1%
34.1%
13.6%
2.1%
0.1%
−3σ
−2σ
−1σ
μ
1σ
2σ
3σ

Properties of normal and log-normal distributions

$$E(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x \cdot \exp\left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx = m$$

$$Var(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-m)^2 \cdot \exp\left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx = \sigma^2$$

If $x = \ln y$ the probability density function of y is given by:

$$\varphi(y) = \frac{1}{\sqrt{2\pi\sigma^2}\, y} \exp\left\{ -\frac{(\ln y - m)^2}{2\sigma^2} \right\} \qquad y > 0$$

Properties of normal and log-normal distributions

This can be seen from $y = e^x$ and $x \sim N(m, \sigma^2)$ and

$$P_Y(y) = P(e^x \le y) = P(x \le \ln(y))$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\ln(y)} \exp\left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx = \int_{-\infty}^{\ln(y)} \varphi(x)dx = \Phi(\ln(y)) - \Phi(-\infty)$$

Now, take the derivative with respect to y:

$$\frac{\partial P_Y}{\partial y} = \frac{\partial \Phi(\ln(y))}{\partial y} = \frac{\partial \ln(y)}{\partial y} \frac{\partial \Phi(\ln(y))}{\partial(\ln(y))} = \frac{1}{y} \varphi(\ln(y))$$

Theorem

For $X \sim N(m, \sigma^2)$ and γ we have

$$E\left[e^{-\gamma X}\right] = \exp\left\{-\gamma m + \frac{1}{2}\gamma^2\sigma^2\right\}$$

proof

$$E\left[e^{-\gamma X}\right] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\gamma x} e^{-\frac{(x-m)^2}{2\sigma^2}}dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[2\gamma x\sigma^2 + x^2 - 2xm + m^2\right]}dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2(m-\gamma\sigma^2)x + m^2\right]}dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[\left(x-(m-\gamma\sigma^2)\right)^2 + 2m\gamma\sigma^2 - \gamma^2\sigma^4\right]}dx$$

$$= e^{-\gamma m + \frac{1}{2}\gamma^2\sigma^2}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[(x-\left[m-\gamma\sigma^2\right])^2\right]}dx$$

$$= e^{-\gamma m + \frac{1}{2}\gamma^2\sigma^2}$$

Theorem

If $X = \ln Y \sim N(m, \sigma^2)$ and $K > 0$, then we have

$$E\left[Y \cdot I_{\{Y>K\}}\right] = e^{m+\frac{1}{2}\sigma^2} N\left(\frac{m-\ln K}{\sigma}+\sigma\right) = E[Y]\, N\left(\frac{m-\ln K}{\sigma}+\sigma\right)$$

<u>Proof</u>: Because $Y > K$, $X > \ln K$, it follows from the definition of the expectation of a random variable that

$$E\left[Y \cdot I_{\{Y>K\}}\right] = E\left[e^X \cdot I_{\{X>\ln K\}}\right] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\ln K}^{\infty} e^x e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\ln K}^{\infty} e^{-\frac{\left(x-(m+\sigma^2)\right)^2}{2\sigma^2}} e^{\frac{2m\sigma^2+\sigma^4}{2\sigma^2}} dx = e^{m+\frac{1}{2}\sigma^2} \int_{\ln K}^{\infty} f_X(x) dx$$

where

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(m+\sigma^2))^2}{2\sigma^2}}$$

is a probability density function for an $N(m + \sigma^2, \sigma^2)$ distributed random variable.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(m+\sigma^2))^2}{2\sigma^2}}$$

$$\int_{\ln K}^{\infty} f_X(x)dx = P(x > \ln K) = P\left(\frac{x - \left[m+\sigma^2\right]}{\sigma} > \frac{\ln K - \left[m+\sigma^2\right]}{\sigma} \right)$$

$$= P\left(-\frac{x - \left[m+\sigma^2\right]}{\sigma} < -\frac{\ln K - \left[m+\sigma^2\right]}{\sigma} \right)$$

$$= N\left(-\frac{\ln K - \left[m+\sigma^2\right]}{\sigma} \right) = N\left(\frac{m - \ln K}{\sigma} + \sigma \right)$$

completes the proof

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(m+\sigma^2))^2}{2\sigma^2}}$$

$$\int_{\ln K}^{\infty} f_X(x)dx = 1 - \int_{-\infty}^{\ln K} f_X(x)dx = P(x < \ln K)$$

$$= 1 - P\left(\frac{x - \left[m + \sigma^2\right]}{\sigma} < \frac{\ln K - \left[m + \sigma^2\right]}{\sigma} \right)$$

$$= 1 - N\left(\frac{\ln K - \left[m + \sigma^2\right]}{\sigma} \right)$$

$$= N\left(-\frac{\ln K - \left[m + \sigma^2\right]}{\sigma} \right) = N\left(\frac{m - \ln K}{\sigma} + \sigma \right)$$

<u>Theorem</u>: If $X = \ln Y \sim N(m, \sigma^2)$ and $K > 0$, then we have

$$E\left[\max\{Y-K,0\}\right] = e^{m+\frac{1}{2}\sigma^2} N\left(\frac{m-\ln K}{\sigma}+\sigma\right) - KN\left(\frac{m-\ln K}{\sigma}\right)$$

$$= E[Y]N\left(\frac{m-\ln K}{\sigma}+\sigma\right) - KN\left(\frac{m-\ln K}{\sigma}\right)$$

<u>Proof</u>: Note that

$$E\left[\max\{Y-K,0\}\right] = E\left[(Y-K)I_{\{Y>K\}}\right] = E\left[Y \cdot I_{\{Y>K\}}\right] - K \cdot \text{Prob}(Y>K)$$

The first term is known from the theorem above. The second term can be rewritten as

$$\text{Prob}(Y>K) = \text{Prob}(X>\ln K) = \text{Prob}\left(\frac{X-m}{\sigma} > \frac{\ln K - m}{\sigma}\right)$$

$$= \text{Prob}\left(\frac{X-m}{\sigma} < -\frac{\ln K - m}{\sigma}\right) = N\left(-\frac{\ln K - m}{\sigma}\right) = N\left(\frac{m-\ln K}{\sigma}\right)$$

The claim now follows immediately.

Martingale

We have mentioned before that a martingale describes a fair game where the profit in average will be zero, even if the gambler is allowed to use previous results on a new stake. A stochastic process $\{X_t\}$ is a martingale if the conditional expectation value of X_t is given by:

$$E[X_t \mid X_u \,;\, u \leq s] = X_s \text{ for all } s < t.$$

For a general definition we need:

1. A probability space $(\Omega, \mathcal{F}, P)$.
2. A filtration, i.e. a sequence of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}$.
3. A stochastic process $X = \{x_k\}$ with random variables $x_0, x_1, \ldots$.

<u>Definition</u>: The process X is **martingale** (MG) if:

i. X is $\underline{\mathcal{F}}$–adapted.

ii. $E[|X(t)|] < \infty \;\forall\; t \geq 0.$

iii. $E[X(t) \mid \mathcal{F}_s] = X(s) \;\;\forall\; s \leq t$ (the martingale property)

Martingale

The meaning that X is $\underline{\mathcal{F}}$–adapted is that all x_k are $\mathcal{F}_k$-measurable. In other words, if we know the information in $\mathcal{F}_k$ then we know the value of x_k. If we in (iii) use ≤ or ≥ instead of = we have a **super**- and a **sub martingale** respectively.

Lemma: If X martingale, then

$$E[\Delta X_n \mid \mathcal{F}_{n-1}] = 0 \quad \forall \; n > 0, \; \Delta X_n = X_n - X_{n-1}.$$

Martingales

Theorem: Under the risk neutral measure Q: $(\tilde{p}, \tilde{q})$, the discounted stock price

$$\left\{(1+r)^{-k} S_k, \mathcal{F}_k\right\}_{k=0}^{n}$$

from the binomial model is martingale.

Proof:

$$E^{Q}\left[(1+r)^{-(k+1)} S_{k+1} \mid \mathcal{F}_k\right] = (1+r)^{-(k+1)} (\tilde{p} \cdot u + \tilde{q} \cdot d) S_k =$$

$$\left(\frac{1}{1+r}\right)^{k+1} \left(\frac{u \cdot (1+r-d)}{u-d} + \frac{d \cdot (u-1-r)}{u-d}\right) S_k = \left(\frac{1}{1+r}\right)^{k+1} \frac{(1+r)(u-d)}{u-d} S_k =$$

$$(1+r)^{-k} S_k$$