Binomial model: numerical algorithm

S_0 C_0	$S_0 \cdot u$ $C_{1,1}$	$S_0 \cdot u^2$ $C_{2,2}$	$S_0 \cdot u^3$ $C_{3,3}$	$S_0 \cdot u^4$ $C_{4,4}$	$S_0 \cdot u^5$ $C_{5,5} = \max\left(X - S_0 \cdot u^5, 0\right)$
	$S_0 \cdot d$ $C_{1,0}$	$S_0 \cdot u \cdot d$ $C_{2,1}$	$S_0 \cdot u^2 \cdot d$ $C_{3,2}$	$S_0 \cdot u^3 \cdot d$ $C_{4,3}$	$S_0 \cdot u^4 \cdot d$ $C_{5,4} = \max\left(X - S_0 \cdot u^4 \cdot d, 0\right)$
		$S_0 \cdot d^2$ $C_{2,0}$	$S_0 \cdot u \cdot d^2$ $C_{3,1}$	$S_0 \cdot u^2 \cdot d^2$ $C_{4,2}$	$S_0 \cdot u^3 \cdot d^2$ $C_{5,3} = \max\left(X - S_0 \cdot u^3 \cdot d^2, 0\right)$
			$S_0 \cdot d^3$ $C_{3,0}$	$S_0 \cdot u \cdot d^3$ $C_{4,1}$	$S_0 \cdot u^2 \cdot d^3$ $C_{5,2} = \max\left(X - S_0 \cdot u^2 \cdot d^3, 0\right)$
				$S_0 \cdot d^4$ $C_{4,0}$	$S_0 \cdot u \cdot d^4$ $C_{5,1} = \max\left(X - S_0 \cdot u \cdot d^4, 0\right)$
					$S_0 \cdot d^5$ $C_{5,0} = \max\left(X - S_0 \cdot d^5, 0\right)$

Binomial model: numerical algorithm

The calculations are performed as follows:

Start at the end of the tree (at time T). The lowest node has the value: $S_0 \cdot d^N$. Set the boundary condition in this node with respect to the option type, (see below).

For each node at the same time, calculate the next price by multiplying with u/d and use the same boundary condition C.

Next, go backward in the tree and calculate all possible stock prices as in the figure above, and thereafter the option value, C. We get:

Binomial model: numerical algorithm

American:

$$C_{4,4} = \max\left\{X - S_0 u^4, e^{-r\cdot\Delta t}\left(P_u \cdot C_{5,5} + P_d \cdot C_{5,4}\right)\right\}$$

$$C_{4,3} = \max\left\{X - S_0 u^3 d, e^{-r\cdot\Delta t}\left(P_u \cdot C_{5,4} + P_d \cdot C_{5,3}\right)\right\}$$

$$C_{4,2} = \max\left\{X - S_0 \cdot u^2 \cdot d^2, e^{-r\cdot\Delta t}\left(P_u \cdot C_{5,3} + P_d \cdot C_{5,2}\right)\right\}$$

$$C_{4,0} = \max\left\{X - S_0 \cdot d^4, e^{-r\cdot\Delta t}\left(P_u \cdot C_{5,1} + P_d \cdot C_{5,0}\right)\right\}$$

European:

$$C_{4,4} = e^{-r\cdot\Delta t} \cdot \left(P_u \cdot C_{5,5} + P_d \cdot C_{5,4}\right)$$

$$C_{4,3} = e^{-r\cdot\Delta t} \cdot \left(P_u \cdot C_{5,4} + P_d \cdot C_{5,3}\right)$$

$$C_{4,0} = e^{-r\cdot\Delta t} \cdot \left(P_u \cdot C_{5,1} + P_d \cdot C_{5,0}\right)$$

Binomial model: The Greeks

$$\Delta = \frac{C_{1,1} - C_{1,0}}{S_0 \cdot u - S_0 \cdot d}$$

$$\Gamma = \frac{\dfrac{C_{2,2} - C_{2,1}}{S_0 \cdot u^2 - S_0 \cdot u \cdot d} - \dfrac{C_{2,1} - C_{2,0}}{S_0 \cdot u \cdot d - S_0 \cdot d^2}}{\dfrac{1}{2} \cdot \left(S_0 \cdot u^2 - S_0 \cdot d^2 \right)}$$

$$\Theta = \frac{C_{2,1} - C_0}{2 \cdot \Delta t}$$

$$\vartheta = \frac{C_0(\sigma) - C_0(\sigma + \Delta\sigma)}{\Delta\sigma}$$

$$\rho = \frac{C_0(r) - C_0(r + \Delta r)}{\Delta r}$$

Binomial model: The Greeks

$$\Delta = \frac{\partial P}{\partial S} \qquad \Gamma = \frac{\partial^2 P}{\partial S^2}$$

$$\upsilon = \frac{\partial P}{\partial \sigma}$$

$$\Theta = \frac{\partial P}{\partial T}$$

$$\rho = \frac{\partial P}{\partial r}$$

Boundary conditions

At maturity we use the following conditions, depending on the option type with strike price *X*

$BC = \max(S - X, 0)$ Call option.

$BC = \max(X - S, 0)$ Put option.

The lowest price is

$$P_{\min}^{put} = X \cdot e^{-r \cdot T} - S \qquad P_{\min}^{call} = S - X \cdot e^{-r \cdot T}$$

Example – American Put Option

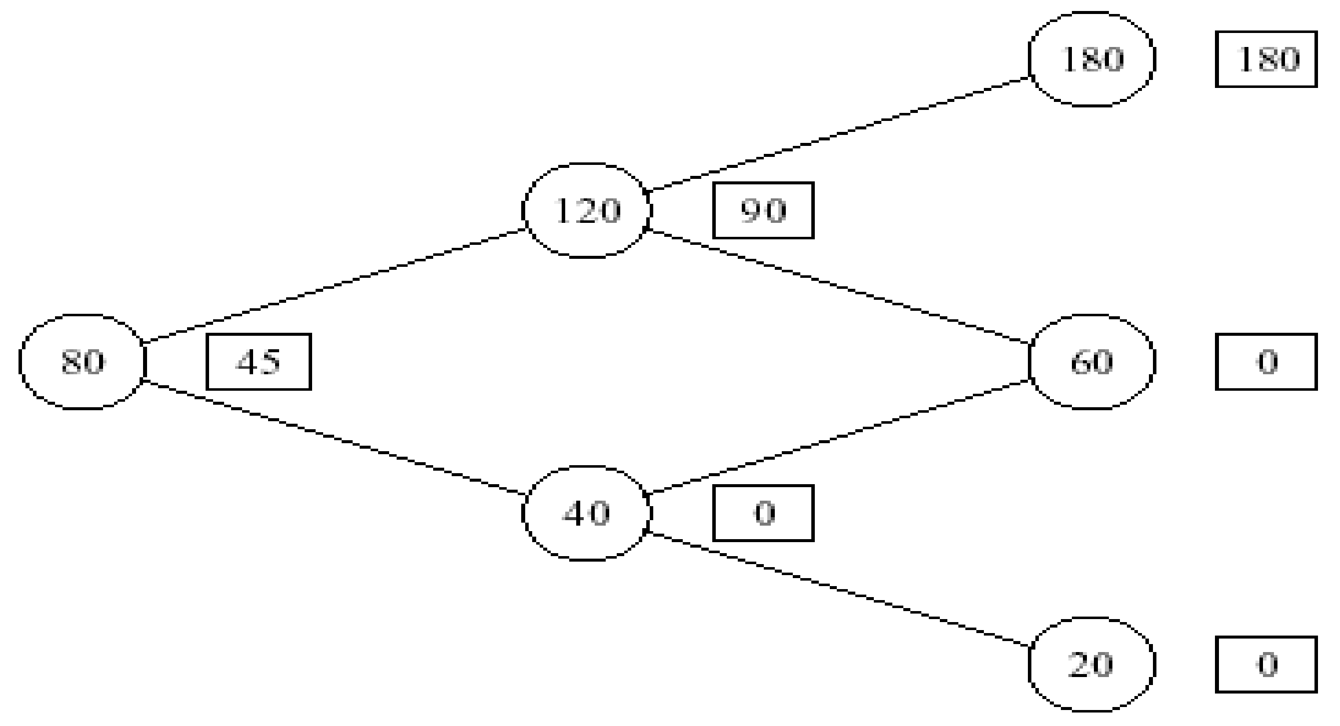
Compute the price of an American put option with strike price $K = 100$ and exercise time $T = 2$ years, using a binomial tree with two trading dates $t_1 = 0$ and $t_2 = 1$ (your portfolio at time $t_3 = 2$ is the same as your portfolio at time $t_2 = 1$) and parameters $s_0 = 100$, $u = 1.4$, $d = 0.8$, $r = 10\%$, and $p = 0.75$

Example – Replicating Portfolio

In the binomial tree below the price of a binary asset-or-nothing option with expiry in two years and payoff

$$X = \begin{cases} S(2) & \text{if } S(2) > 120 \\ 0 & \text{otherwise} \end{cases}$$

has been computed using the parameters $s_0 = 80$, $u = 1.5$, $d = 0.5$, $r = 0$, and $p = 0.50$. In the definition of the contract function $S(2)$ denotes the stock price at time $t = 2$.



Find the replicating portfolio for this option and verify that the option is self-financing.

Replicating Portfolio

We can use the values to calculate the replicating portfolio. At $t = 0$ the following must hold:

$$\begin{cases} x + y \cdot 120 = 90 \\ x + y \cdot 40 = 0 \end{cases}$$

Since regardless if the stock price goes up or down the value of the portfolio should equal the value of the option. This yields: $x = -45$ and $y = 9/8$. We can also use:

$$\begin{cases} x = \dfrac{1}{1+r}\dfrac{u \cdot \Phi(d) - d \cdot \Phi(u)}{u-d} = \dfrac{1}{1}\dfrac{1.5 \cdot 0 - 0.5 \cdot 90}{1.5 - 0.5} = -45 \\ y = \dfrac{1}{S_0}\dfrac{\Phi(u) - \Phi(d)}{u-d} = \dfrac{1}{80}\dfrac{90-0}{1.5-0.5} = \dfrac{9}{8} \end{cases}$$

The same calculations can be made to find the replicated portfolio in all the nodes, e.g., where $S = 120$:

$$\begin{cases} x = \dfrac{1}{1}\dfrac{1.5 \cdot 0 - 0.5 \cdot 180}{1.5 - 0.5} = -90 \\ y = \dfrac{1}{120}\dfrac{180-0}{1.5-0.5} = \dfrac{3}{2} \end{cases}$$

Probabilities in the model

$$S_{\max} = S_0 \cdot u^n = S_0 \cdot e^{n\cdot\sigma\cdot\sqrt{dt}} \qquad u = e^{\sigma\cdot\sqrt{dt}}$$

$$S_{\min} = S_0 \cdot d^n = S_0 \cdot e^{-n\cdot\sigma\cdot\sqrt{dt}} \qquad d = e^{-\sigma\cdot\sqrt{dt}}$$

$$P(S_{\max}) = P_u^{\,n} \quad resp. \quad P(S_{\min}) = P_d^{\,n}$$

					5, 5	1 path
				4, 4		
			3, 3		5, 4	5 paths
		2, 2		4, 3		
	1, 1		3, 2		5, 3	10 paths
0, 0		2, 1		4, 2		
	1, 0		3, 1		5, 2	10 paths
		2, 0		4, 1		
			3, 0		5, 1	5 paths
				4, 0		
					5, 0	1 path
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	

Finite difference methods

Parabolic boundary value problem of the Black-Scholes type :

$$-\frac{\partial C}{\partial t} = \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta) S \frac{\partial C}{\partial S} - rC$$

If we let $x = \ln(S)$ we can rewrite the PDE by use of:

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial C}{\partial x}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial C}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial C}{\partial x} + \frac{1}{S}\frac{\partial}{\partial x}\frac{\partial C}{\partial S} = -\frac{1}{S^2}\frac{\partial C}{\partial x} + \frac{1}{S}\frac{\partial}{\partial x}\left(\frac{1}{S}\frac{\partial C}{\partial x}\right)$$

$$= -\frac{1}{S^2}\frac{\partial C}{\partial x} + \frac{1}{S^2}\frac{\partial^2 C}{\partial x^2}$$

Finite difference methods

$$-\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2} - \frac{1}{2}\sigma^2 \frac{\partial C}{\partial x} + (r - \delta)\frac{\partial C}{\partial x} - rC$$

$$-\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2} + \upsilon \frac{\partial C}{\partial x} - rC$$

where $v = r - \delta - ½\sigma^2$

By doing this we have removed the explicit dependencies of *S* and thereby get the coefficients independent of the stock price!!!

The explicit finite difference method

$$\frac{\partial C}{\partial x} = \frac{C_{i+1,j+1} - C_{i+1,j-1}}{2 \cdot \Delta x}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{C_{i+1,j+1} - 2 \cdot C_{i+1,j} + C_{i+1,j-1}}{\Delta x^2}$$

Backward differences

$$-\frac{C_{i+1,j} - C_{i,j}}{\Delta t} = \frac{1}{2}\sigma^2 \frac{C_{i+1,j+1} - 2 \cdot C_{i+1,j} + C_{i+1,j-1}}{\Delta x^2} + \nu \cdot \frac{C_{i+1,j+1} - C_{i+1,j-1}}{2 \cdot \Delta x} - r \cdot C_{i+1,j}$$

$$C_{i,j} = \frac{1}{1 + r \cdot \Delta t}\left(p_u \cdot C_{i+1,j+1} + p_m \cdot C_{i+1,j} + p_d \cdot C_{i+1,j-1} \right)$$

$$p_u = \frac{1}{2} \cdot \Delta t \cdot \left(\frac{\sigma^2}{\Delta x^2} + \frac{\nu}{\Delta x} \right)$$

$$p_m = 1 - \Delta t \cdot \frac{\sigma^2}{\Delta x^2}$$

$$\Delta x \geq \sigma \sqrt{3 \cdot \Delta t}$$

$$p_d = \frac{1}{2} \cdot \Delta t \cdot \left(\frac{\sigma^2}{\Delta x^2} - \frac{\nu}{\Delta x} \right)$$

The implicit finite difference method

$$\frac{\partial C}{\partial x} = \frac{C_{i,j+1} - C_{i,j-1}}{2 \cdot \Delta x}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{C_{i,j+1} - 2 \cdot C_{i,j} + C_{i,j-1}}{\Delta x^2}$$

Forward differences

$$-\frac{C_{i+1,j} - C_{i,j}}{\Delta t} = \frac{1}{2} \cdot \sigma^2 \cdot \frac{C_{i,j+1} - 2 \cdot C_{i,j} + C_{i,j-1}}{\Delta x^2} + \nu \cdot \frac{C_{i,j+1} - C_{i,j-1}}{2 \cdot \Delta x} - r \cdot C_{i+1,j}$$

$$p_u \cdot C_{i,j+1} + p_m \cdot C_{i,j} + p_d \cdot C_{i,j-1} = C_{i+1,j}$$

$$p_u = \frac{1}{2} \cdot \Delta t \cdot \left(\frac{\sigma^2}{\Delta x^2} + \frac{\nu}{\Delta x} \right)$$

$$p_m = 1 + \Delta t \cdot \frac{\sigma^2}{\Delta x^2} + r \cdot \Delta t$$

$$p_d = \frac{1}{2} \cdot \Delta t \cdot \left(\frac{\sigma^2}{\Delta x^2} - \frac{\nu}{\Delta x} \right)$$

boundary conditions

$$C_{i,N_j} - C_{i,N_{j-1}} = \lambda_U$$

$$C_{i,-N_{j+1}} - C_{i,-N_j} = \lambda_L$$

call

$$\lambda_U = S_{i,N_j} - S_{i,N_{j-1}}$$

$$\lambda_L = 0$$

put

$$\lambda_U = 0$$

$$\lambda_L = S_{i,-N_j} - S_{i,-N_{j+1}}$$

The implicit finite difference method

$$
\begin{bmatrix}
1 & -1 & 0 & \dots & \dots & \dots & 0 \\
p_u & p_m & p_d & 0 & \dots & \dots & 0 \\
0 & p_u & p_m & p_d & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & p_u & p_m & p_d & 0 \\
0 & \dots & \dots & 0 & p_u & p_m & p_d \\
0 & \dots & \dots & \dots & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
C_{i,N_j} \\
C_{i,N_{j-1}} \\
C_{i,N_{j-2}} \\
\dots \\
C_{i,-N_{j+2}} \\
C_{i,-N_{j+1}} \\
C_{i,-N_j}
\end{bmatrix}
=
\begin{bmatrix}
\lambda_U \\
C_{i+1,N_{j-1}} \\
C_{i+1,N_{j-2}} \\
\dots \\
C_{i+1,N_{j+2}} \\
C_{i+1,N_{j+1}} \\
\lambda_L
\end{bmatrix}
$$

Crank-Nicholson

$$-\frac{C_{i+1,j}-C_{i,j}}{\Delta t}=\frac{1}{2}\cdot\sigma^2\cdot\left(\frac{\left(C_{i+1,j+1}-2\cdot C_{i+1,j}+C_{i+1,j-1}\right)+\left(C_{i,j+1}-2\cdot C_{i,j}+C_{i,j-1}\right)}{2\Delta x^2}\right)$$

$$+\nu\cdot\left(\frac{\left(C_{i+1,j+1}-C_{i+1,j-1}\right)+\left(C_{i,j+1}-C_{i,j-1}\right)}{4\cdot\Delta x}\right)-r\cdot\left(\frac{C_{i+1,j}+C_{i,j}}{2}\right)$$

$$p_u\cdot C_{i,j+1}+p_m\cdot C_{i,j}+p_d\cdot C_{i,j-1}=-p_u\cdot C_{i+1,j+1}-\left(p_m-2\right)\cdot C_{i+1,j}+p_d\cdot C_{i+1,j-1}$$

$$p_u=-\frac{1}{4}\cdot\Delta t\cdot\left(\frac{\sigma^2}{\Delta x^2}+\frac{\nu}{\Delta x}\right)$$

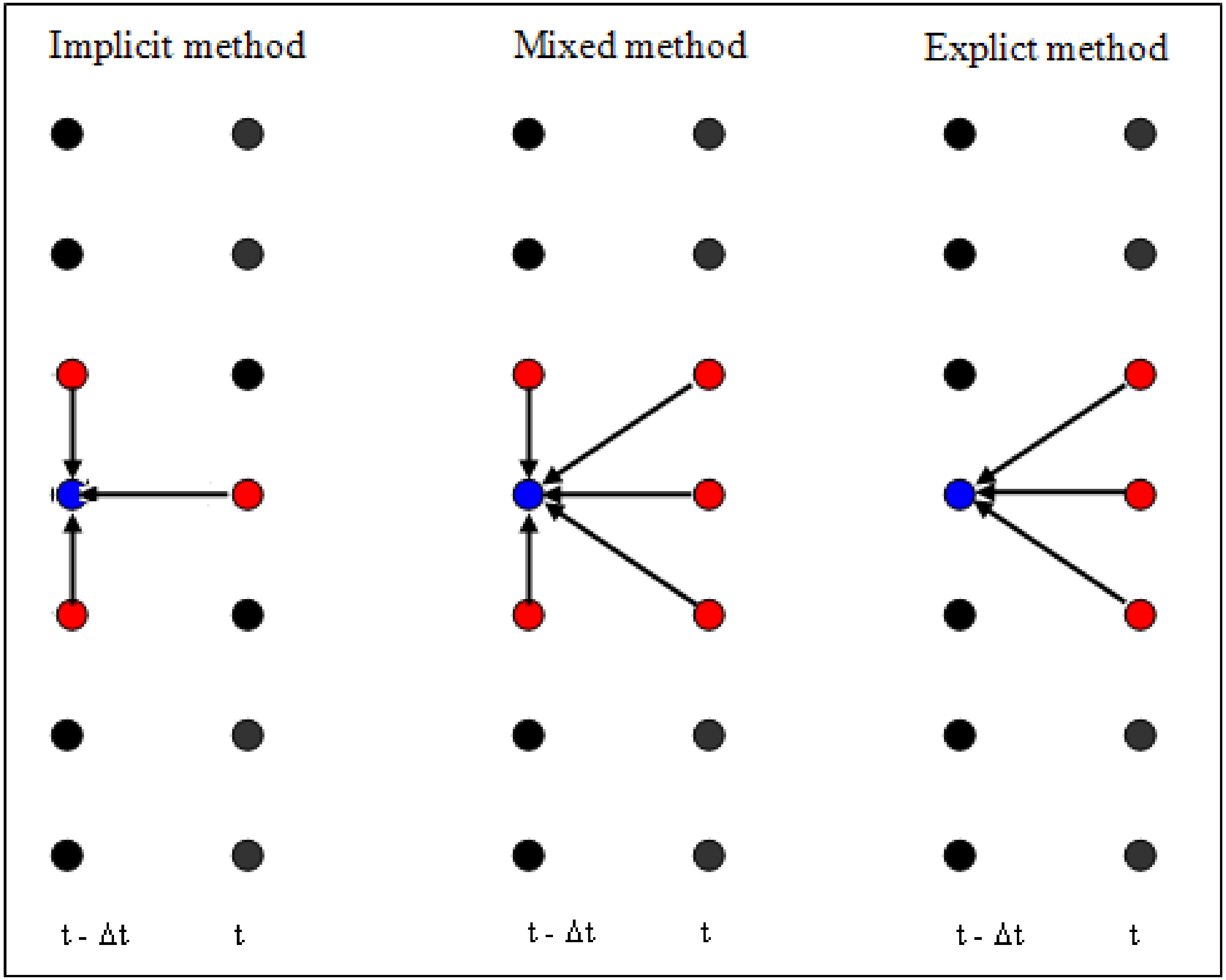
$$p_m=1+\Delta t\cdot\frac{\sigma^2}{2\cdot\Delta x^2}+\frac{r\cdot\Delta t}{2}$$

$$p_d=-\frac{1}{4}\cdot\Delta t\cdot\left(\frac{\sigma^2}{\Delta x^2}-\frac{\nu}{\Delta x}\right)$$

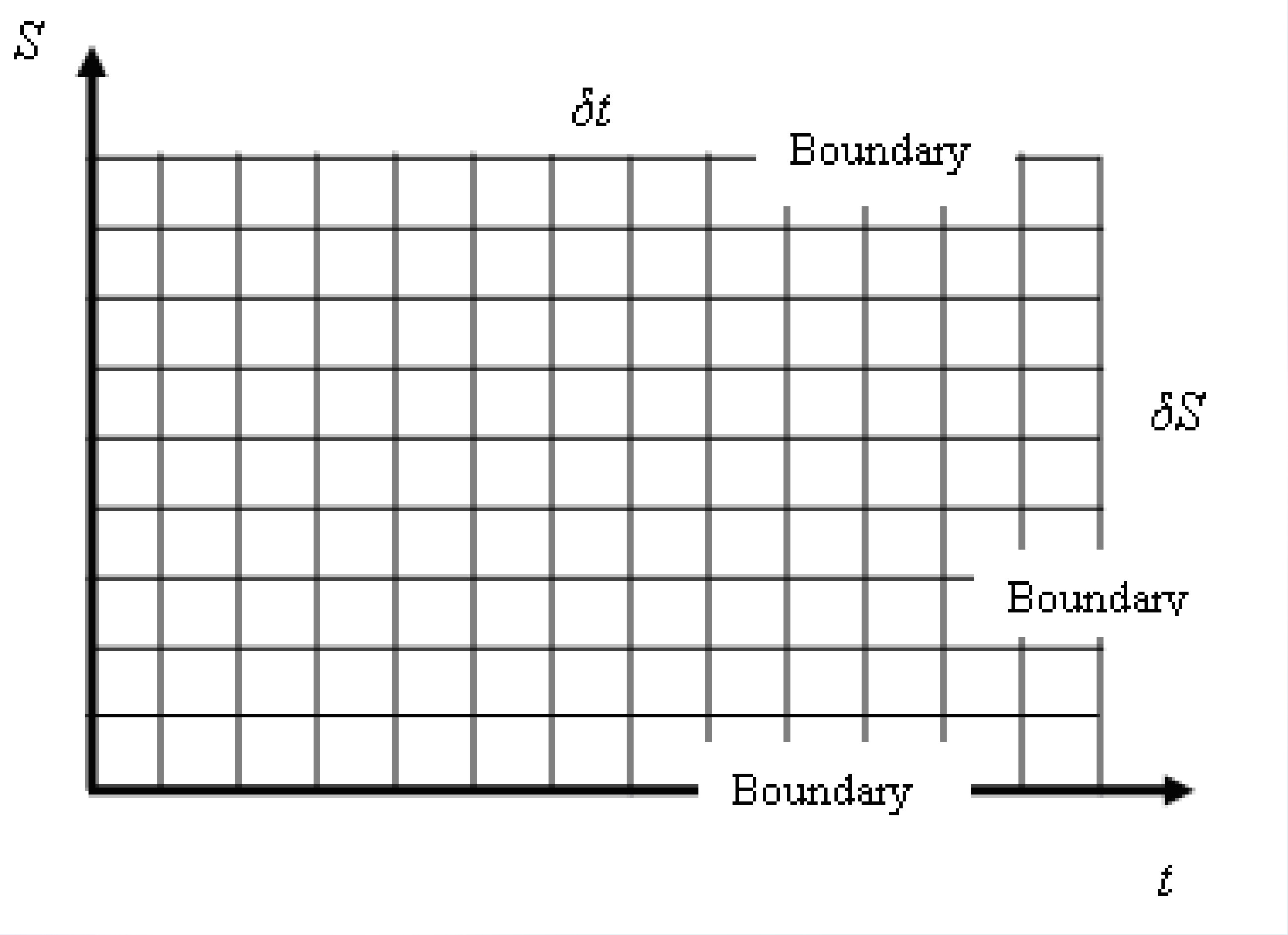
The accuracy in the methods above are:

$O(\Delta x + \Delta t)$, $O(\Delta x^2 + \Delta t)$ and $O(\Delta x^2 + (\Delta t/2)^2)$ respectively.

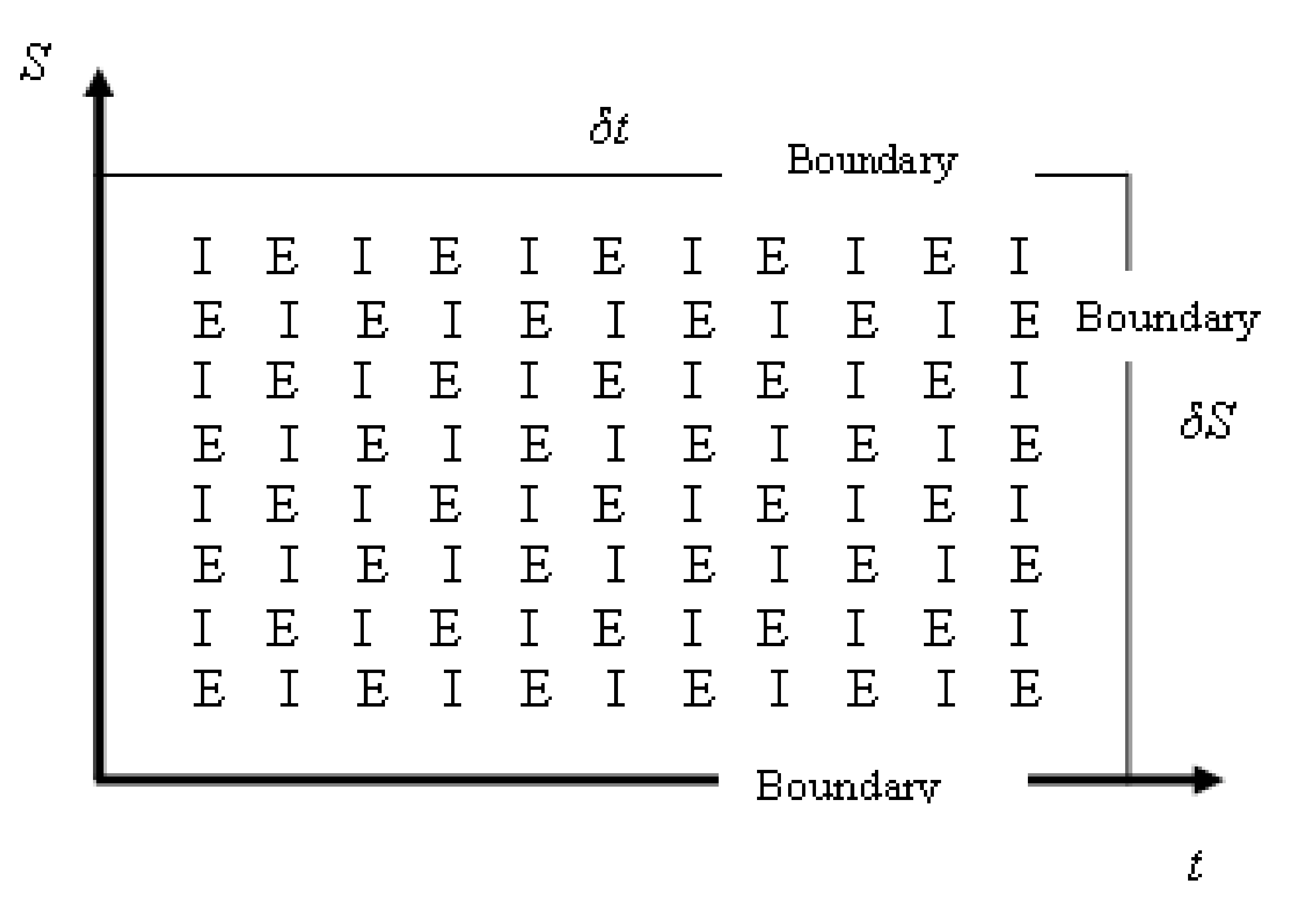
Schema - Finite Difference



The Hopscotch method



The Hopscotch method



Monte-Carlo Simulations

The stock price is simulated by a stochastic process:

$$dS_t = rS_t dt + \sigma S_t dz_t$$

For simplicity, study the natural logarithm of the stock price: $x_t = \ln(S_t)$ which gives:

$$dx_t = \nu dt + \sigma dz_t$$

$$\nu = r - \frac{1}{2}\sigma^2$$

➔

$$x_{t+\Delta t} = x_t + \nu\Delta t + \sigma\left(z_{t+\Delta t} - z_t\right) \qquad z_{t+\Delta t} - z_t = \sqrt{\Delta t} \cdot \varepsilon$$

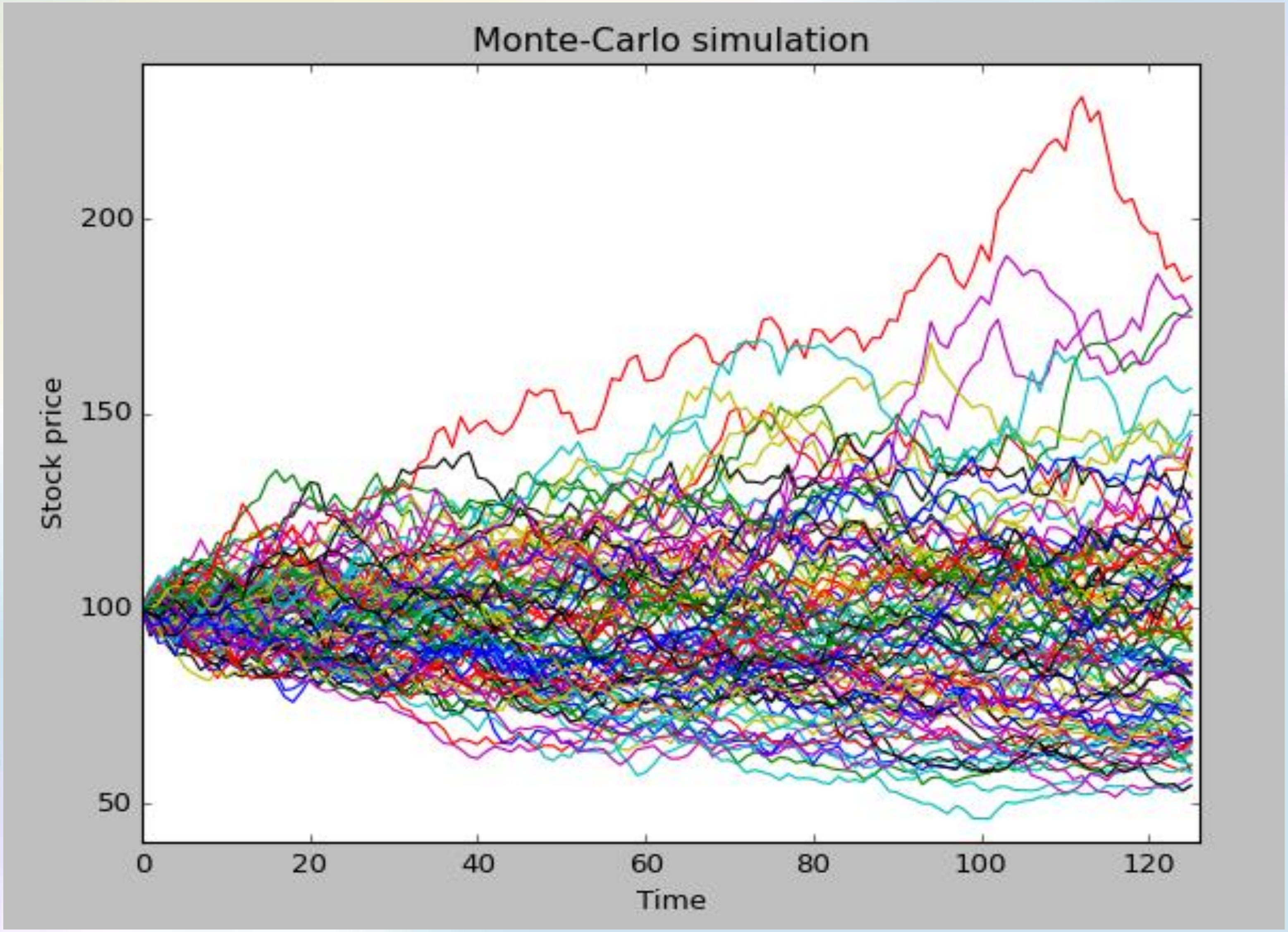
➔

$$\begin{cases} x_{t_i} = x_{t_{i-1}} + \nu\Delta t + \sigma\sqrt{\Delta t} \cdot \varepsilon \\ S_{t_i} = \exp\left(x_{t_i}\right) \end{cases}$$

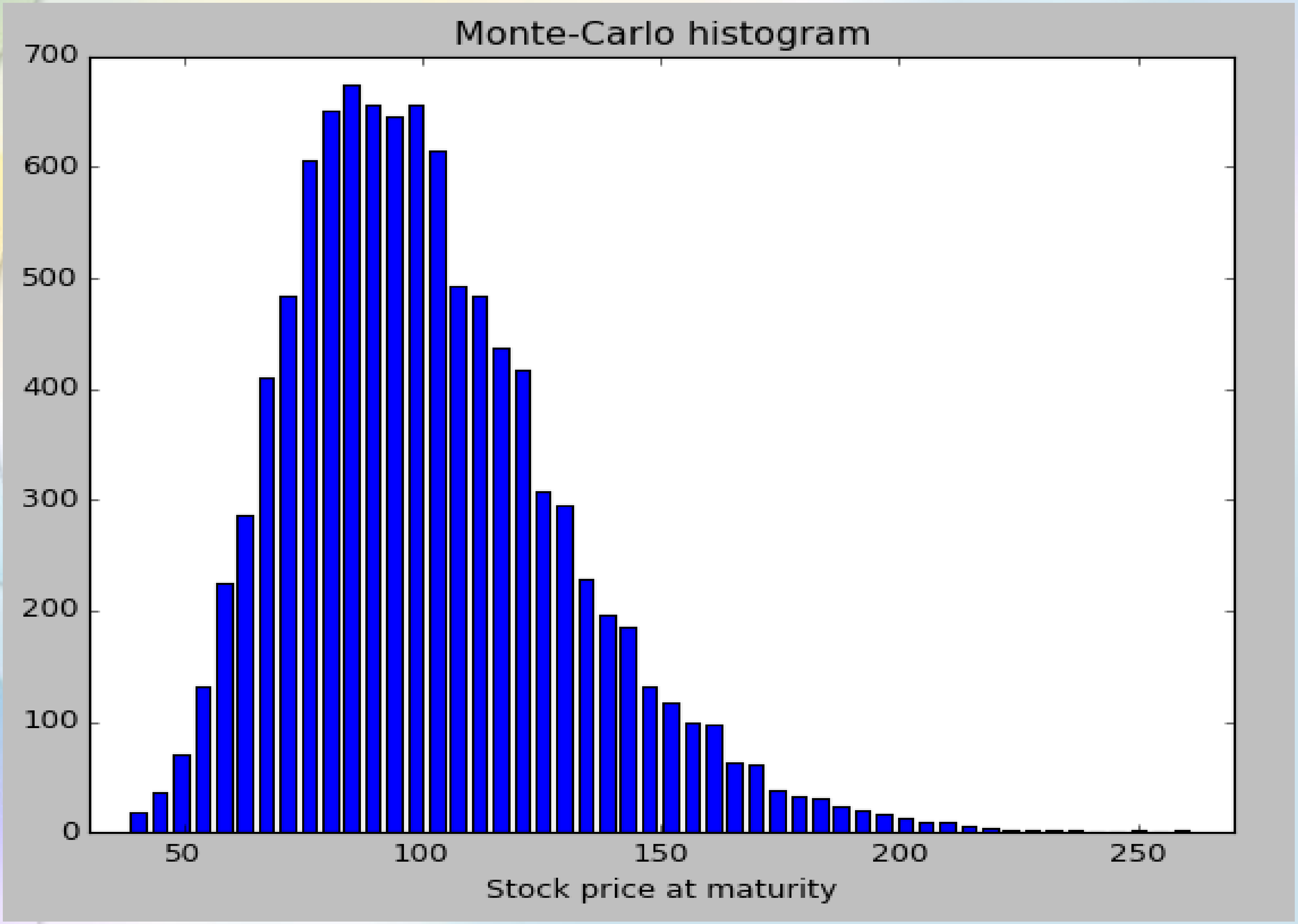
Monte-Carlo Simulations

In the figure below, we show 100 such simulations of the stock price during a half of a year divided into 100 intervals. At the starting time, the stock price is 100, the volatility is 40% and the risk-free interest rate 6%.

Monte-Carlo Simulations



Monte-Carlo Simulations (10 000)



Monte-Carlo Simulations

For each scenario, we then calculate the profit of the call options as: $\max(S_T - X, 0)$. To find the theoretical option value we calculate the mean value of the discounted pay-off:

$$C_0 = \exp(-rT)\frac{1}{N}\sum_{i=1}^{N}\max\left(S_{T,i} - X, 0\right)$$

where X is the strike price of the option. The standard deviation (SD) and the standard error (SE) of the simulations is given by: (Remember: the annualized volatility σ is the standard deviation of the instrument's yearly logarithmic returns.)

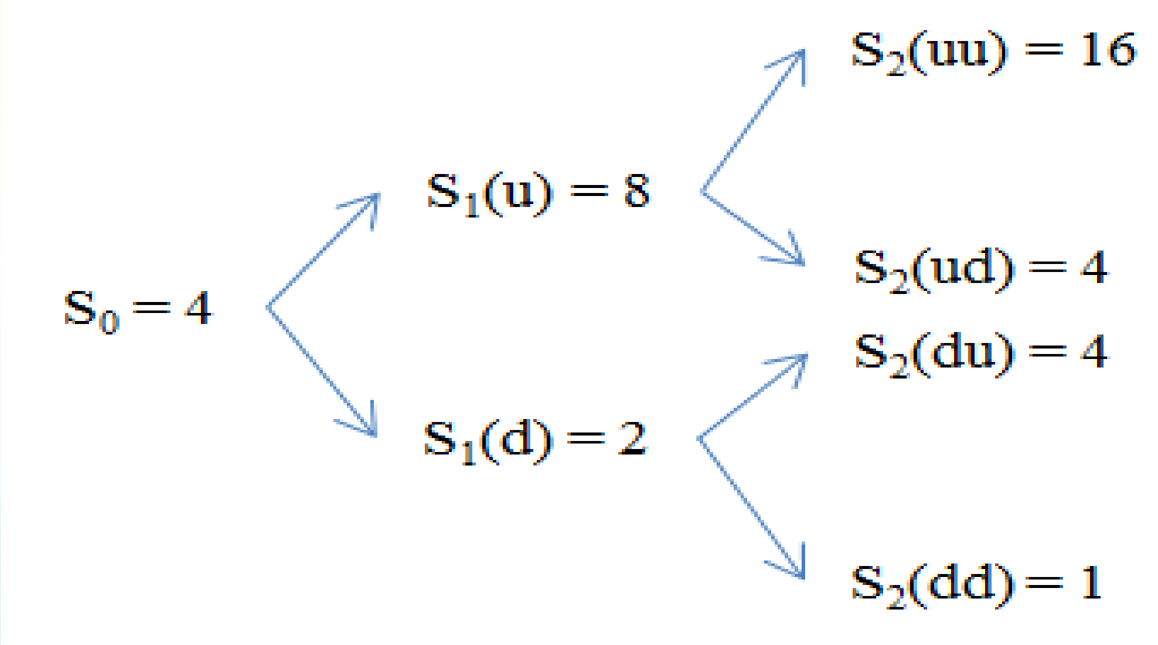
$$SD = \frac{\sigma}{\sqrt{N}} = \frac{1}{N-1}\sqrt{\sum_{i=1}^{N}\left(C_{T,i}\right)^2 - \frac{1}{N}\left(\sum_{i=1}^{N}C_{T,i}\right)^2} \cdot \exp\left(-2rT\right)$$

$$SE = \frac{SD}{\sqrt{N}}$$

Introduction to probability theory

Study a Binomial three with the following properties:

$u = 2 \Rightarrow d = 1/u = 0.5, S_0 = 4$ and $P_u = P_d = ½$.



where $S2(uu) = u^2 S_0, \quad S2(ud) = udS_0, ...$

Introduction to probability theory

If we are tossing a coin one, two and tree times, we get the following **sample space**:

$\Omega_1 = \{u, d\} = \{\omega_1\}$,

$\Omega_2 = \{uu, ud, du, dd\} = \{\omega_2\}$,

$\Omega_3 = \{uuu, uud, udu, duu, udd, dud, ddu, ddd\} = \{\omega_3\}$

Introduce the **interest rate *r***: 1 CU (cash unit) -> $(1 + r) \cdot 1$ CU $= 1 \cdot R$ CU. The factor R must be in the interval: $d \leq R \leq u$ because if $R > u$ nobody should be interested to buy the stock, if $R < d$ then $r < 0$ which is unrealistic.

<u>Statement</u>: We say that the model above is free of arbitrage if: $d \leq R \leq u$.

Introduction to probability theory

Example: Let us study a European call option with strike K at $t = 1$. On maturity, the value is given by:

$$V_1(\omega) = (S_1(\omega) - K)^+ = \max(S_1(\omega) - K, 0)$$

We are looking for the arbitrage-free price. The two possible outcome, u and d are given by:

$$V_1(\omega) = \begin{cases} (uS_0 - K)^+ & \text{if} \quad \omega_1 = u \\ (dS_0 - K)^+ & \text{if} \quad \omega_1 = d \end{cases}$$

To hedge a short position of the option we have to buy Δ_0 stocks. At $t = 0$ we have then sold the option, giving us V_0 cash units. But we also buy Δ_0 stocks at S_0. We then have $(V_0 - \Delta_0 S_0)$ cash units to put in the bank (or that's what we had to borrow, depending of the sign) at a rate of r, where $R = 1 + r$. The value process gives us two possible values on maturity:

Introduction to probability theory

$$V_1(u) = \Delta_0 \cdot S_1(u) + R \cdot (V_0 - \Delta_0 \cdot S_0)$$
$$V_1(d) = \Delta_0 \cdot S_1(d) + R \cdot (V_0 - \Delta_0 \cdot S_0)$$

We get

$$\Delta_0 = \frac{V_1(u) - V_1(d)}{S_1(u) - S_1(d)} \quad \rightarrow \quad \frac{\partial V}{\partial S}$$

By inserting Δ_0 into the equation above, we find the price of the option at $t = 0$:

$$V_0 = \frac{1}{R}\left\{\frac{R-d}{u-d} V_1(u) - \frac{R-u}{u-d} V_1(d)\right\} = \frac{1}{R}\{q_u \cdot V_1(u) + q_d \cdot V_1(d)\} = \frac{1}{R} E^Q\left[V_1\right]$$

where we have defined *p* and *q* as the **risk-neutral probabilities**:

$$q_u = \frac{R-d}{u-d} \qquad q_d = -\frac{R-u}{u-d}$$

Introduction to probability theory

We let the expression

$$\Pi[X] = \frac{1}{R} E^Q[X]$$

represent the arbitrage free price on the option on the contingent claim X with respect to the risk-neutral probability measure Q, the martingale measure. Similar, we get

$$V_1(u) = \frac{1}{R}\{p \cdot V_2(uu) + q \cdot V_2(ud)\}; \qquad \Delta_1(u) = \frac{V_2(uu) - V_2(ud)}{S_2(uu) - S_2(ud)}$$

$$V_1(d) = \frac{1}{R}\{p \cdot V_2(du) + q \cdot V_2(dd)\}; \qquad \Delta_1(d) = \frac{V_2(du) - V_2(dd)}{S_2(du) - S_2(dd)}$$

and so on…..

Finite Probability Spaces

Let $\mathcal{F}$ be the set of all subsets to the sample space:

Ω (Ø, {*ddd*}, {*uuu*, *uud*, *udu*, *ddd*},

Ω are examples of some) where Ø is the empty set. Then, we define a **probability measure** P by a function mapping $\mathcal{F}$ into the interval [0, 1] with $P(\Omega) = 1$, where

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P\left(A_k\right)$$

and $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$. Probability measures has the following interpretation: Let A be a subset of $\mathcal{F}$. Imagine that Ω is the set of all possible outcomes of some random experiment. There is a certain probability, between 0 and 1, that when that experiment is performed, the outcome will lie in the set A. We think of $P(A)$ as this probability. From now we will use $P_u = 1/3$ and $P_d = 2/3$.

σ-algebra

Definition: A **σ-algebra** is a collection $\mathcal{F}$ of sub sets in Ω with the following three properties:

$$
\begin{cases}
\emptyset \in \mathcal{F} \\
A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F} \\
A_1, A_2 \ldots \text{ is a sequence of subspaces to } \mathcal{F} \Rightarrow \bigcup_k A_k \in \mathcal{F}
\end{cases}
$$

It is essential to understand that, in probabilistic terms, the σ-algebra can be interpreted as "containing all relevant information" about a random variable.

σ-algebra

Example: Some important σ-algebras to Ω above is:

$\mathcal{F}_0 = \{\emptyset, \Omega\}$

$\mathcal{F}_1 = \{\emptyset, \Omega, \{uuu, uud, udu, udd\}, \{duu, dud, ddu, ddd\}\}$

$F_2 = \{\emptyset, \Omega, \{uuu, uud\}, \{udu, udd\}, \{duu, dud\}, \{ddu, ddd\}\}$ and all unions of these}

$\mathcal{F}_3 = \mathcal{F}$ = the set of all sub sets of Ω.

We say that $\mathcal{F}_3$ is **finer** than $\mathcal{F}_2$, which is finer than $\mathcal{F}_1$.

If we introduce the terms $A_u = \{uuu, uud, udu, udd\} = \{u**\}$,

$A_d = \{d**\}$,

$A_{uu} = \{uu*\}$ etc, we can write:

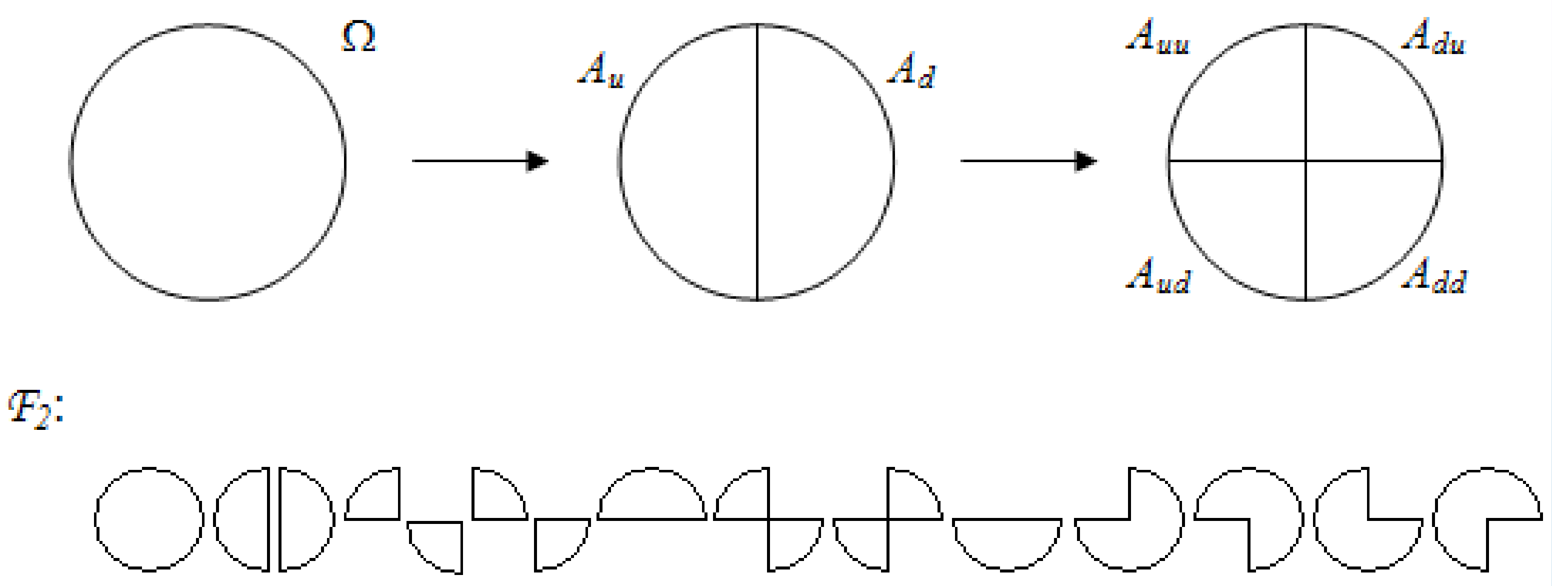
$\mathcal{F}_1 = \{\emptyset, \Omega, A_u, A_d\}$

$\mathcal{F}_2 = \{\emptyset, \Omega, A_u, A_d, A_{uu}, A_{ud}, A_{du}, A_{dd}, A_{uu} U A_{du}, A_{uu} U A_{dd}, A_{ud} U A_{du}, A_{ud} U A_{dd}, A_{uu}^c, A_{ud}^c, A_{du}^c, A_{dd}^c\}$

Filtrations

$\mathcal{F}_1 = \{\varnothing, \Omega, A_u, A_d\}$

$\mathcal{F}_2 = \{\varnothing, \Omega, A_u, A_d, A_{uu}, A_{ud}, A_{du}, A_{dd}, A_{uu} U A_{du}, A_{uu} U A_{dd}, A_{ud} U A_{du}, A_{ud} U A_{dd}, A_{uu}{}^c, A_{ud}{}^c, A_{du}{}^c, A_{dd}{}^c\}$



Measures

Definition: A pair (X, $\mathcal{F}$), where X is a set and $\mathcal{F}$ an σ-algebra on X is called a **measurable space**. The sub-spaces that exist in $\mathcal{F}$ are called **$\mathcal{F}$-measurable** sets.

In particular, if a random variable Y is a function of X, $Y = \Phi(X)$, then Y is $\mathcal{F}^X$-measurable.

Definition: A **finite measure μ** on a measurable space is a function such as:

$(i)\quad \mu(A) \geq 0,$

$(ii)\quad \mu(\emptyset) = 0,$

(iii) If $A_k \in \mathcal{F}\ \forall\ k = 1, 2, \ldots$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

Some definitions

Definition: A **filtration** $\mathcal{F}_\infty = \underline{\mathcal{F}} = \{\mathcal{F}_t;\ t \geq 0\}$ is a sequence of σ-algebras $\mathcal{F}_0$, $\mathcal{F}_1$, ..., $\mathcal{F}_n$ such that $\mathcal{F}_t$ contains all sets in $\mathcal{F}_{t-1}$:

$$\begin{cases} \mathcal{F}_t \subseteq \mathcal{F} & \forall t \geq 0 \\ s \leq t \quad \Rightarrow & \mathcal{F}_s \in \mathcal{F}_t \end{cases}$$

If we consider a finite probability space $(\Omega, \mathcal{F}_t, P)$ with the filtration of σ-algebras sometimes called **σ-fields**.

Definition: X is $\underline{\mathcal{F}}$**-adapted** if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Definition: A function f: $X \rightarrow \boldsymbol{R}$ is said to be $\mathcal{F}$ **–measurable** if for each interval I the set $f^{-1}(I)$ is $\mathcal{F}$ –measurable, i.e.:

$$\{x \in X \mid f(x) \in I\} \in \mathcal{F}$$

Definition: A **stochastic variable** X is a mapping of Ω on R:

$X : \Omega \text{ -> } R$ so that X is $\mathcal{F}$-measurable

Stochastic Process

Definition: A **stochastic process** can be considered as a discrete set of time indexed random variables or, as in time, a continuous set . In many situations we consider such a process containing a **drift** μ and **diffusion** σ:

$$X(t + \Delta t) - X(t) = \mu[t, X(t)]\, \Delta t + \sigma[t, X(t)] Z(t)$$

Sometimes this is interpreted as a random process (a random walk) upon a deterministic drift. In the continuous limit the random process becomes a Wiener process.

Wiener Process

Definition: A stochastic process $\{W(t);\ t \geq 0\}$ is called a **Wiener process** if:

1. $W(0) = 0$
2. $(W(u) - W(t))$ and $(W(s) - W(r))$ are independent (i.e. W have independent increments) $r \leq s \leq t \leq u$.
3. $W(t) - W(s)$ is normal distributed $N\left[0,\sqrt{t-s}\right]$ ". $t \leq s$
4. $W(t)$ have continuous trajectories.

A very important property of a Wiener process (a Brownian motion) is $(dW)^2 = dt$.

In risk neutral valuation, we have a risk-free bond and a stock following the process:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)$$

Expectation value

Definition: The **expectation value** (or mean value) of X given $(\Omega, \mathcal{F}, P)$ is:

$$E[X] = \sum_{\omega \in \Omega} X(\omega) P\{\omega\}$$

in the discrete case and

$$E[X] = \int_{\Omega} X(\omega) dP\{\omega\}$$

in the continuous case.

Variance

<u>Definition</u>: The **Variance of X**:

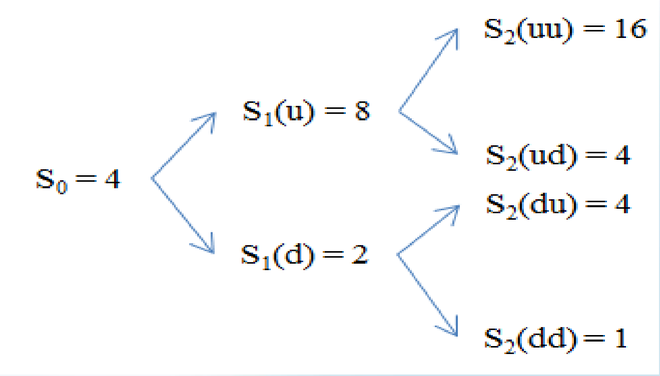
$$Var[X] = \int_{\Omega} \left[X(\omega) - E[X] \right]^2 dP\{\omega\}$$

$$Var[X] = \sum_{\omega=\Omega} \left(X(\omega) - E[X(\omega)] \right)^2 P\{\omega\} = \sum_{k=1}^{n} \left(x_k - E[X(\omega)] \right)^2 \mu_X(x_k) =$$
$$= E\left[\left(X(\omega) - E[X(\omega)] \right)^2 \right] = E\left[X^2(\omega) \right] - \left(E[X(\omega)] \right)^2$$

Example

<u>Example</u>: Calculate $E[S_3]$

$$E\left[S_3\right] = S_2(uuu)P\left\{uuu\right\} + S_2(uud)P\left\{uud\right\} + S_2(udu)P\left\{udu\right\} + S_2(udd)P\left\{udd\right\} +$$
$$S_2(duu)P\left\{duu\right\} + S_2(ddu)P\left\{ddu\right\} + S_2(dud)P\left\{dud\right\} + S_2(ddd)P\left\{ddd\right\}$$
$$= 16 \cdot P(A_{uu}) + 4 \cdot P(A_{ud} \cup A_{du}) + P(A_{dd})$$
$$= 16 \cdot P\left\{S_2 = 16\right\} + 4 \cdot P\left\{S_2 = 4\right\} + P\left\{S_2 = 1\right\}$$
$$= 16 \cdot \mu_{S_2}\left\{16\right\} + 4 \cdot \mu_{S_2}\left\{4\right\} + \mu_{S_2}\left\{1\right\} = 16 \cdot \frac{1}{9} + 4 \cdot \frac{4}{9} + \frac{4}{9} = \frac{36}{9} = 4$$



Indicator function

Definition: An **indicator function** *I* defined by:

$$I_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

where *A* is called **a set indicated by** I_A.

Definition: A function *h* is called **simple** if

$$h(x) = \sum_{k=1}^{n} c_k I_k(x)$$

Probability spaces

A **Probability spaces is defined by (Ω, $\mathcal{F}$, *P*), *where:***

- Ω is a non empty set, sample space, which contains all possible outcomes of some random experiment.
- $\mathcal{F}$ *is a σ-algebra of all subsets of* Ω.
- *P is a probability measure on* (Ω, $\mathcal{F}$) *which assigns to each set* $A \in \mathcal{F}$ *a* number *P(A) = [0, 1], which represent the probability that the outcome of the* random experiment lies in *A*.

Given (Ω, $\mathcal{F}$, *P*) *and a stochastic variable X. If X is a indicator function* (e.g., $X(\Omega) = I_A(\Omega) = 1$ *if* $\omega \in A$ *and 0 otherwise) then:*

$$\int_{\Omega} XdP = P(A)$$

If *X is simple*

$$\int_{\Omega} XdP = \sum_{k=1}^{n} c_k \int_{\Omega} I_{A_k} dP = \sum_{k=1}^{n} c_k P(A_k) \qquad \int_{A} XdP = \int_{\Omega} X \cdot I_A dP$$