Exotic options

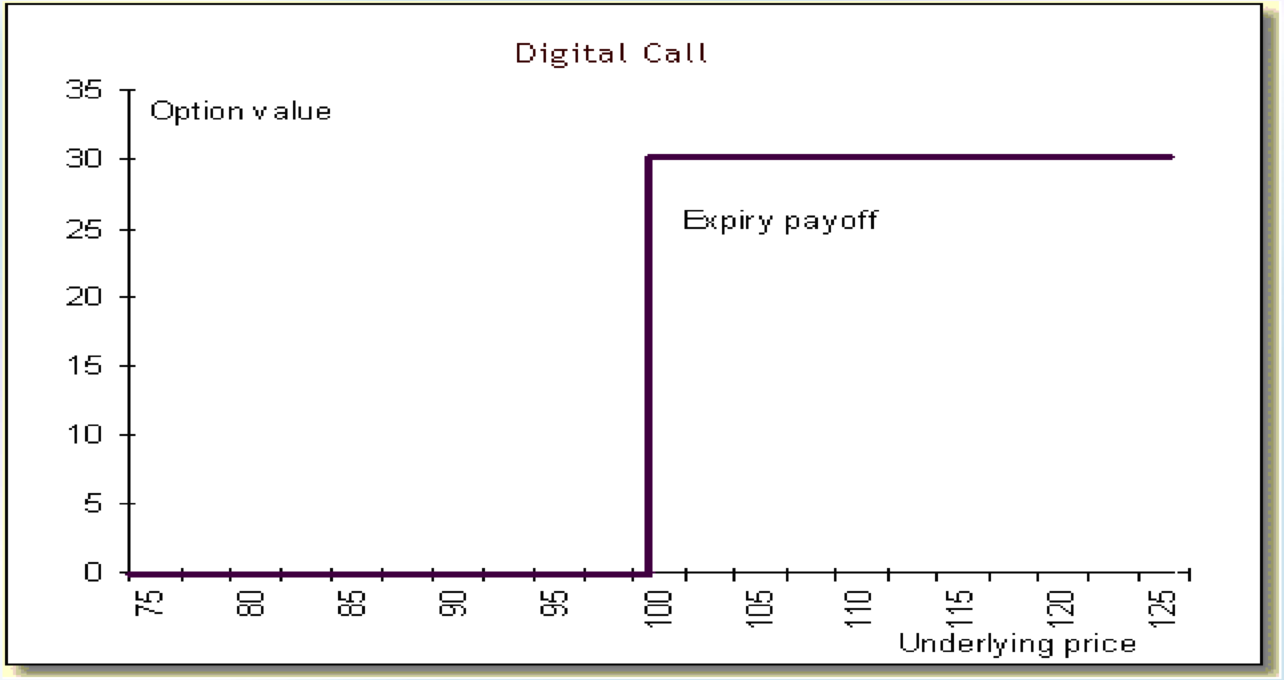
A standard option has some well-defined properties:

Type	Call or Put
Style	European, Bermudan or American
Strike	A given price X
Expiry date	The time of maturity
Settlement type	Physical or Cash delivery
Underlying	Stock, currency, index, etc

For exotic options one or some of the above properties are defined differently or additional properties are added.

Digital (Binary) options

Two kind, **cash-or-nothing** and **asset-or-nothing options**. **Cash-or-nothing digital options** pay a fixed amount of cash if they expire in the money, otherwise nothing They are also known as: **Binary options**, **All-or-nothing options** or **Bet options**.



Digital options - pricing

Cash-or-nothing

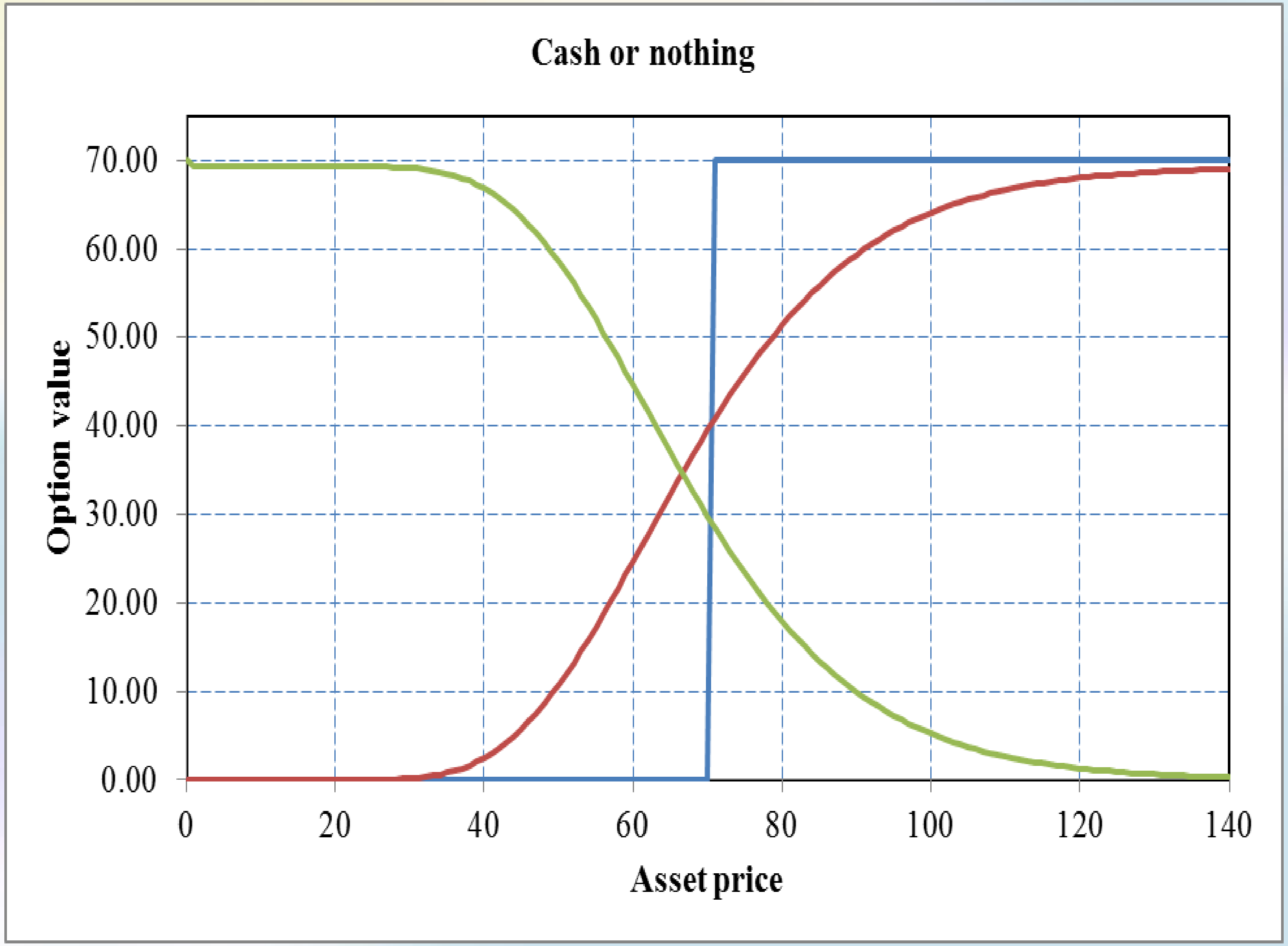
$$P_{call} = e^{-rT} KN(d) \qquad P_{put} = e^{-rT} KN(-d)$$

$$d = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma \cdot \sqrt{T}}$$

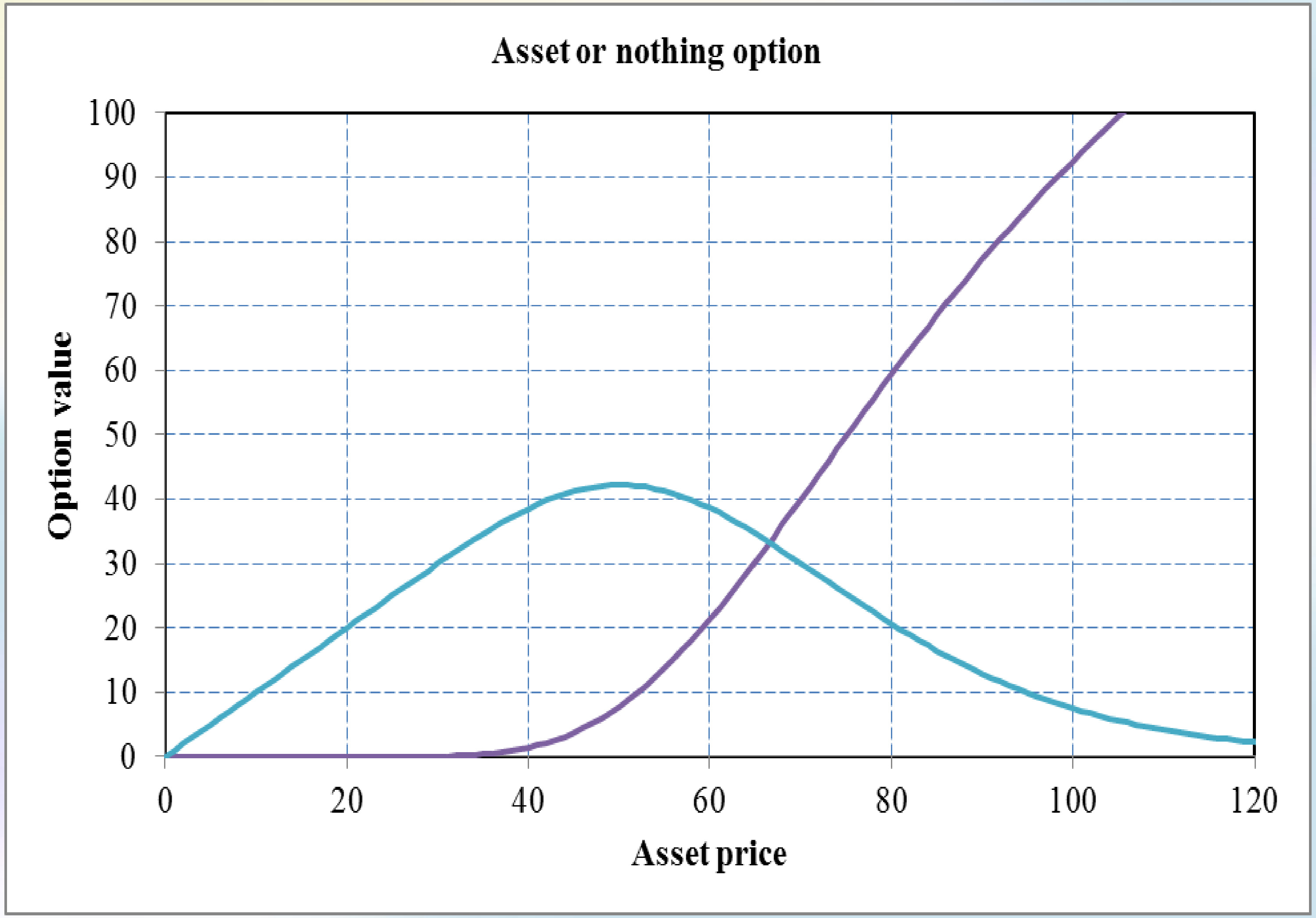
Asset-or-nothing

$$P_{call} = e^{-qT} SN(d) \qquad P_{put} = e^{-qT} SN(-d)$$

Cash or nothing



Asset or noting



Supershare

Payoff: 0 if $X_L > S > X_H$ and S/X_L otherwise

$$P = \frac{e^{-rT}}{X_L}\left[N(d_1) - N(d_2)\right]$$

$$d_1 = \frac{\ln\left(\frac{S}{X_L}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma \cdot \sqrt{T}}; \qquad d_2 = \frac{\ln\left(\frac{S}{X_H}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma \cdot \sqrt{T}}$$

American digitals

Reiner and Rubenstein derived formula for American digitals in 1991. Sometimes they are called one-touch binary/digital or binary-at-hit. The value of a call option of one-touch-down and one-touch-up digital is given by:

$$P_{one\_touch\_down} = K \cdot \left[\left(\frac{H}{S} \right)^{\mu+\lambda} N(z) + \left(\frac{H}{S} \right)^{\mu-\lambda} N\left(z - 2 \cdot \lambda \cdot \sigma\sqrt{T} \right) \right]$$

$$P_{one\_touch\_up} = K \cdot \left[\left(\frac{H}{S} \right)^{\mu+\lambda} N(-z) + \left(\frac{H}{S} \right)^{\mu-\lambda} N\left(2 \cdot \lambda \cdot \sigma\sqrt{T} - z \right) \right]$$

$$z = \frac{\ln\left(\frac{H}{S} \right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

$$\mu = \frac{r - q - \frac{1}{2}\sigma^2}{\sigma^2}$$

$$\lambda = \sqrt{\mu^2 + \frac{2(r-q)}{\sigma^2}}$$

Gap options

Asset-or-nothing digitals (**gap options**) are a combination, a conventional option and a cash-or-nothing digital. A gap option is the right to buy (for a call) or sell (for a put) an asset at time $T > 0$ for a price $G > 0$ if the asset exceeds (for a call) or falls below a price $X > 0$. It is straightforward to write down the price for a gap call as:

$$C_g(S,T,G,X) = e^{-rT} E^Q\left[S(T) - G \,|\, S(T) > X\right]$$

A gap call can also be written as a portfolio with a usual call and a digital:

$$C_g(S,T,G,X) = e^{-\overline{rT}}\left(E^Q\left[S(T) - X \,|\, S(T) > X\right] - E^Q\left[G - X \,|\, S(T) > X\right]\right)$$

$$C_g(S,T,G,X) = C(S,T,X) - (G - X)\,C_d(S,T,X)$$

For an example, see the lecture notes

Collars

A collar is an option to buy an asset at strike price $X > 0$, but the total payoff is capped at $Z > X$.
The claim at time T is $\min((S(T) - X)^+, (S(T) - Z)^+)$, which equals $(S(T) - X)^+ - (S(T) - Z)^+$.

Thus, a collar can be priced as:

$$C_c(S, T, X, Z) = C(S, T, X) - C(S, T, Z)$$

Knock-out and Knock-in Options

The payoff of a conventional option depends only on the price of the underlying relative to the strike at the time of exercise, but there are so-called **path-dependent options** whose payoffs also depend on the *history* of the underlying price. One class of this type is the barrier option.

There exist two general classes of barrier options; in-options and out-options. With in-options the buyer get an option that becomes active if and when the underlying hit a given barrier value.

An out-option is an option, which is active from the beginning, but becomes inactive, i.e., expires immediately if the underlying hits the barrier value.

It is possible to combine both types. If we have a down-and-out-call option and a down-and-in-call option and the underlying hit the barrier, the down-and-out becomes inactive while the down-and-in becomes active. Therefore this combination is an exact replication of a plain vanilla European call option.

Down-and-out put option

$$P = Xe^{-rT}\left\{N(d_4) - N(d_2) - a\left[N(d_7) - N(d_5)\right]\right\} -$$
$$S_0\left\{N(d_3) - N(d_1) - b\left[N(d_8) - N(d_6)\right]\right\}$$

$$a = \left(\frac{S_b}{S_0}\right)^{-1+2r/\sigma^2}, \qquad b = \left(\frac{S_b}{S_0}\right)^{1+2r/\sigma^2}$$

$$d_1 = \frac{\log(S_0/X) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\log(S_0/X) + (r-\sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_3 = \frac{\log(S_0/S_b) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_4 = \frac{\log(S_0/S_b) + (r-\sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_5 = \frac{\log(S_0/S_b) - (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_6 = \frac{\log(S_0/S_b) - (r-\sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_7 = \frac{\log(S_0X/S_b^2) - (r-\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_8 = \frac{\log(S_0X/S_b^2) - (r+\sigma^2/2)T}{\sigma\sqrt{T}},$$

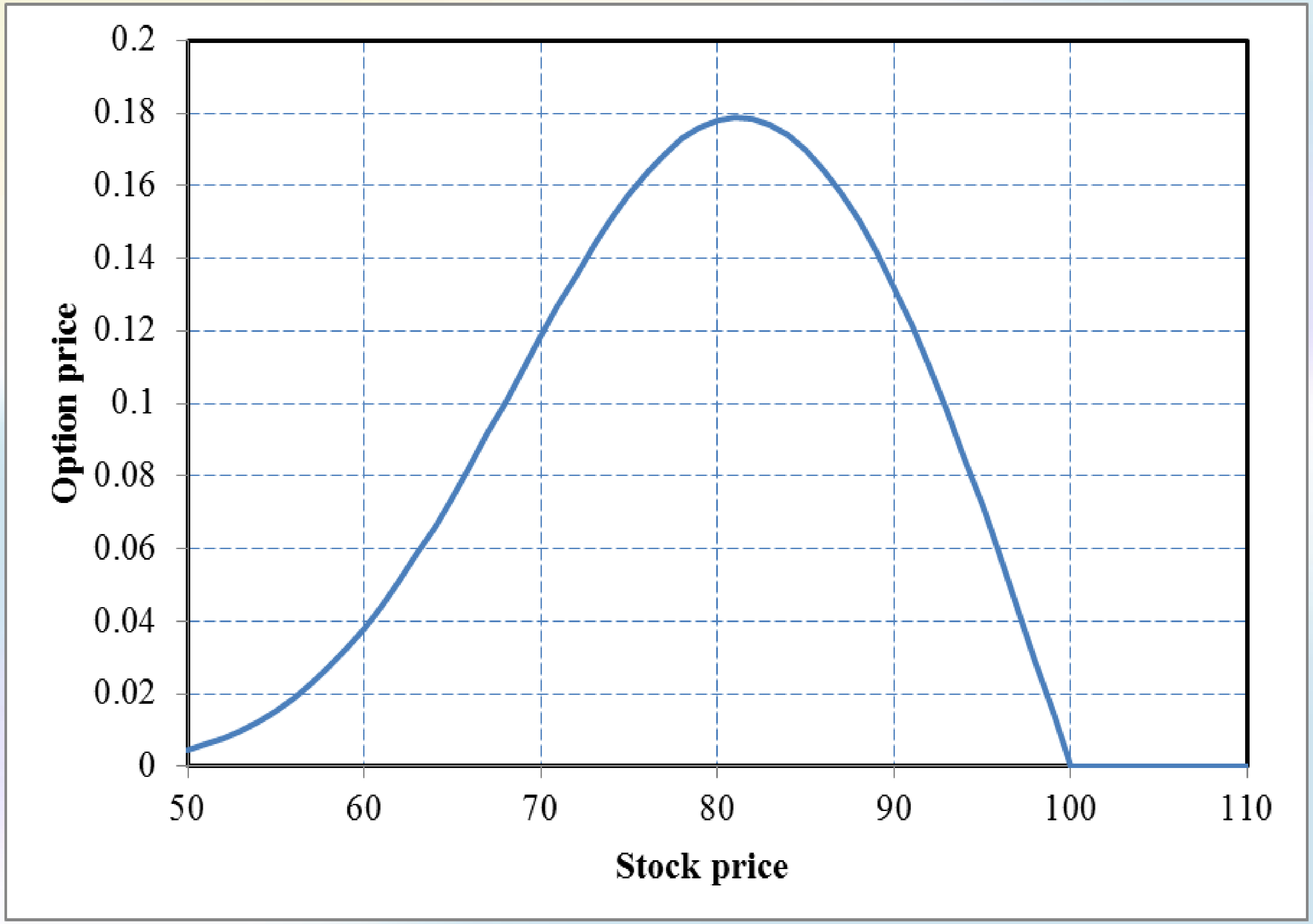
Knock-out and Knock-in Options

Some contracts have more than one barrier - e.g. a double knock-out option knocks out if either a higher or a lower barrier is reached

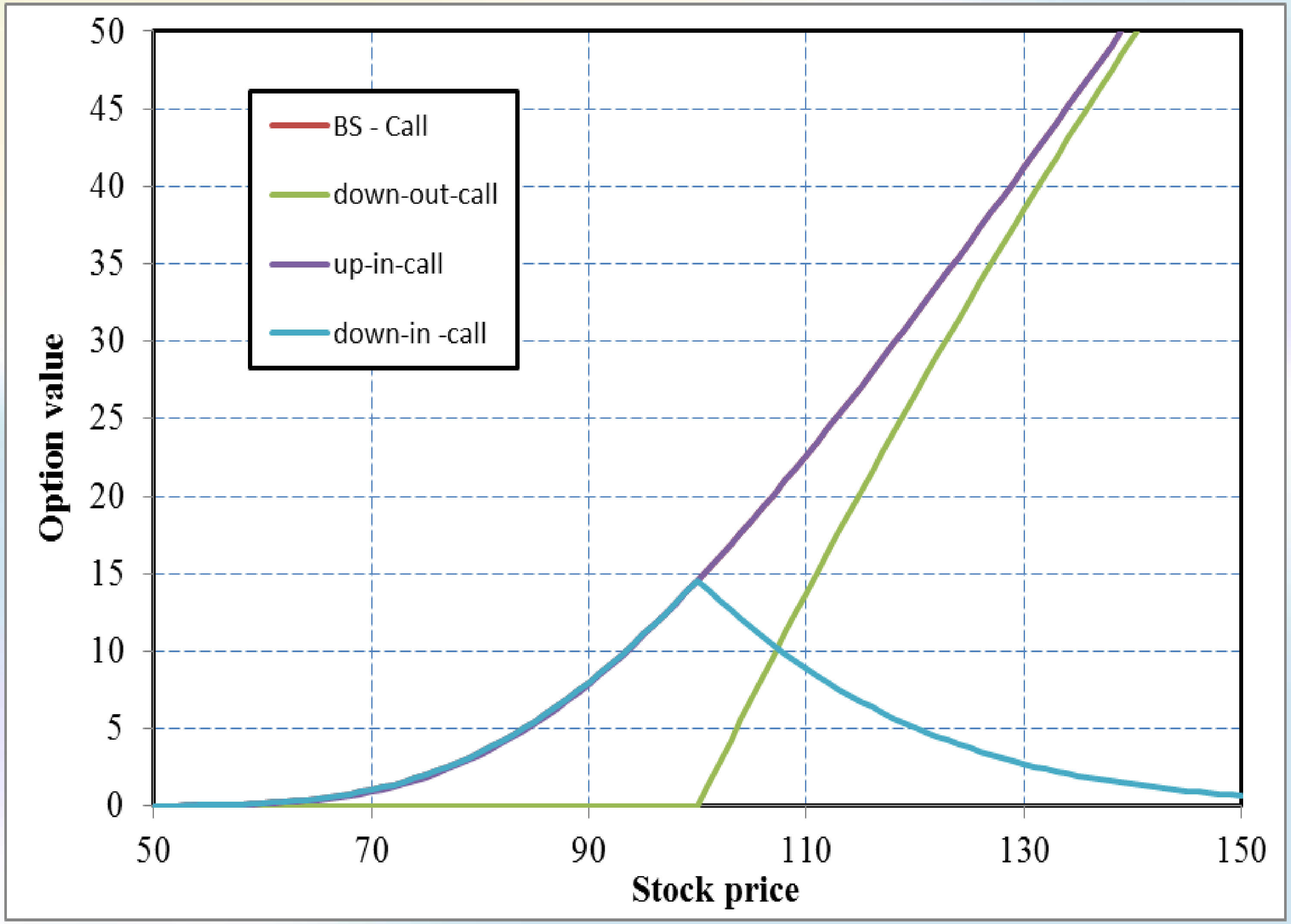
Some barrier options knock in or out depending on the performance of a ***different*** market. An example of this type is the **soft call provision** embedded in many Euroconvertible bonds, which gives the issuer the right to call the bond if the underlying ***shares*** reach a specified threshold level.

See the Lecture notes!

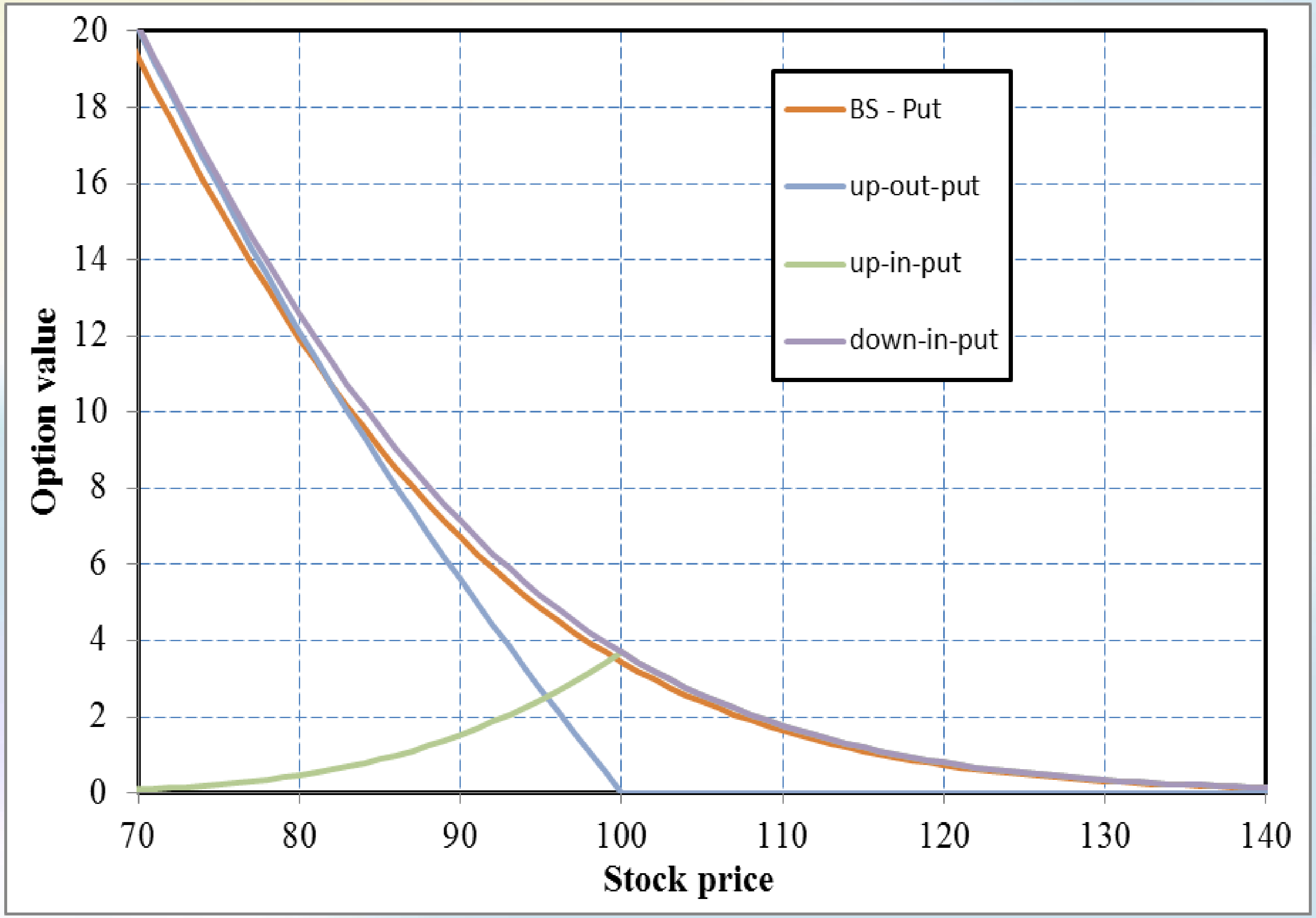
An up and out call option with strike price 90 and a barrier level 100



Barrier options and a plain vanilla Call



Barrier options and a plain vanilla Put



Lookback Options

A floating strike lookback call option gives the holder the right to buy the underlying security to the lowest observed value S_{min}, during the option lifetime.

$$P_{call} = SN(a_1) - S_{\min} e^{-rT} N\left(a_2\right)$$

$$+ Se^{-rT} \frac{\sigma^2}{2r} \left[\left(\frac{S}{S_{\min}} \right)^{-\frac{2r}{\sigma^2}} N(-a_1 + \frac{2r}{\sigma}\sqrt{T}) - e^{-rT} N(-a_1) \right]$$

$$a_1 = \frac{\ln\left(\frac{S}{S_{\min}}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma \cdot \sqrt{T}}, \qquad a_2 = a_1 - \sigma \cdot \sqrt{T}$$

Lookback Options

Similarly, the holder of a floating strike lookback put options have the right to sell the underlying security to the highest observed price S_{max}, during the life time of the option.

$$P_{put} = S_{\max} e^{-rT} N(-b_2) - SN\left(-b_1\right)$$
$$+ Se^{-rT} \frac{\sigma^2}{2r}\left[-\left(\frac{S}{S_{\max}}\right)^{-\frac{2r}{\sigma^2}} N(b_1 - \frac{2r}{\sigma}\sqrt{T}) + e^{rT} N(b_1) \right]$$

$$b_1 = \frac{\ln\left(\frac{S}{S_{\max}}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma \cdot \sqrt{T}}, \qquad b_2 = b_1 - \sigma \cdot \sqrt{T}$$

Lookback Options

A fixed strike lookback call option gives the holder the maximum difference between the price and the strike during a given period. Also other types of lookback options can be constructed (see Haug).

Asian Options

Asian options are especially popular on the currency- and commodity market. An average value option is less volatile than the underlying itself. Therefore, the price of an average-rate option is lower than a plain vanilla option. Options based on an average value are more stable and they are more difficult to be manipulated in price by the underlying.

Asian options come in two basic flavours:

- **Average price options**
- **Average strike options**

A mean value option

We will construct an exotic European option where the holder at the day of maturity T_2 receives:

$$X = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u) du$$

where $T_1 < T_2$ is fix. Calculate the arbitrage-free price. We know that

$$\begin{cases} dS(t) = r \cdot S(t) \cdot dt + \sigma \cdot S(t) \cdot dW \\ S(T) = s \end{cases}$$

Integrate

$$S(t) = s + r \int_0^t S(u) \cdot du + \sigma \int_0^t S(u) \cdot dW(u)$$

$$\Pi[X \mid \mathcal{F}] = e^{-r(T_2 - t)} E_{t,s}^Q \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u) du \right] = \frac{e^{-r(T_2 - t)}}{T_2 - T_1} \int_{T_1}^{T_2} E_{t,s}^Q [S(u)] du$$

A mean value option - 2

$$E\left[S(t)\right] = s + r\int_0^t E\left[S(u)\right]\cdot du + 0$$

Let $E[S(t)] = m$ and take the derivative

$$\begin{cases} \dot{m}(t) = r\cdot m(t) \\ m(0) = s \end{cases}$$

The solution is given by

$$m(t) = E\left[S(t)\right] = se^{rt}$$

$$\Pi\left[X \mid \mathcal{F}\right] = \frac{s\cdot e^{-r(T_2-t)}}{T_2 - T_1}\int_{T_1}^{T_2} e^{r(u-t)}du = \frac{s/r}{T_2 - T_1}\cdot\left(1 - e^{-r(T_2-T_1)}\right)$$

Asian Options

There are three main approaches to pricing Asian options:

European-style options based on geometric averages can be priced by adapting the analytical models. This is because if the underlying price is assumed to be log-normally distributed then its geometric average is also log-normal. The formulas below is given by Kemna and Vorst (1990)

$$P_{call} = Se^{(b-r)T} N(d_1) - Ke^{-rT} N(d_2)$$

$$P_{put} = Ke^{-rT} N(-d_2) - Se^{(b-r)T} N(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(b + \frac{\sigma_A^2}{2}\right)T}{\sigma_A \cdot \sqrt{T}}, \qquad d_2 = d_1 - \sigma_A \cdot \sqrt{T}$$

$$b = \frac{1}{2}\left(r - \frac{\sigma^2}{6}\right) \qquad \sigma_A = \frac{\sigma}{\sqrt{3}}$$

Asian Options

There is no equivalent solution for options based on arithmetic averages, because even if the underlying price is log-normally distributed, the arithmetic average is not. However, various analytic approximations have been developed which work reasonably well. A weak approximation by Turnbull and Wakeman is given in the book of Haug.

Any Asian option, no matter what its style or averaging method, may be priced by Monte Carlo simulation, but this is computationally much more intensive.

The volatility of an average is always less than that of the price itself, and the longer the averaging period the lower is its volatility.

Whatever pricing method is used, average price options come out very much cheaper than conventional ones. The option price’s sensitivity to spikes in the underlying market is reduced, hence also its price.

Chooser Options

A simple chooser option gives the holder the right to chose if the option will become a call- or a put option after a certain time t_1. This is also known as U-choose option. The strike price K, is the same for both options and also the maturity T_2. In addition to all the standard terms of a conventional option, the chooser includes a clause that specifies the choose date - the date by which the buyer must tell the seller whether the option is to be a call or a put. After this date the option becomes a conventional call or put.

$$P = SN(d) - Ke^{-rT_2} N\left(d - \sigma\sqrt{T_2}\right) - SN(-y) + Ke^{-rT_2} N\left(-y - \sigma\sqrt{t_1}\right)$$

$$d = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T_2}{\sigma \cdot \sqrt{T_2}}, \qquad y = \frac{\ln\left(\frac{S}{K}\right) + rT_2 + \frac{\sigma^2}{2} t_1}{\sigma \cdot \sqrt{t_1}}$$

Chooser Options

A complex chooser option was introduced by Rubenstein (1991), where the strike was not the same for the call and the put options.
If a call expires at time T_1 with the strike X_1 and a put expires at T_2 with strike X_2 the claim of the chooser option is max$\{C(S, T_1 - T, X_1), P(S, T_2 - T, X_2)\}$.

Consider the special case: $T_1 = T_2 = \tau$ and $X_1 = X_2 = X$, the solution is simplified. Applying the Put-Call parity, we can rewrite the claim at T as

$$\Phi(T) = \max\left\{C(S,\tau - T, X), C(S,\tau - T, X) + Xe^{-r(\tau - T)} - S(T)\right\}$$
$$= C(S,\tau - T, X) + \left[Xe^{-r(\tau - T)} - S(T)\right]^+$$

This is equivalent to the claim of a portfolio consisting of a call and a put.
Thus the price of the chooser option is given by:

$$CO(S,T,X) = C(S,\tau - T, X) + P(S,T, Xe^{-r(\tau - T)})$$

Chooser Options - 2

For the general case where $T_1 \neq T_2$ or $X_1 \neq X_2$, the composition is a little bit more complicated. Let S^* be the solution S to

$$C(S, T_1 - T, X_1) = P(S, T_2 - T, X_2).$$

When $S(T) > S^*$, the put becomes worthless and the value of the chooser option becomes that of the call. When $S(T) < S^*$, the call option becomes worthless and the value of the chooser option becomes that of the put. Hence, the claim of the chooser option at time T can be written as

$$\Phi(T) = \left[C(S, T_1 - T, X_1)\right]^+ + \left[P(S, T_2 - T, X_2)\right]^+$$

which implies that the claim of a chooser option at time T can be replicated with a call on call and a call on put, each with a zero strike price. Thus, the price of a general chooser is:

$$CO(S, T_1, T_2, X_1, X_2) = CC(S, T, T_1, 0, X_1) + CP(S, T, T_2, 0, X_2)$$

Forward Options

In a forward option, the strike is set at some specified future date, rather than on the effective date. This is also known as: **Forward-start options** or **Delayed-start options**. The actual strike will not be known until the future **effective date**, but the contract does specify what the strike will be in relation to the underlying market price.

Assumed that the strike price of a forward-start option expiring at time τ equals $\alpha S(T)$, where $\alpha > 0$ is a constant and $T < [0, \tau)$ is the issue time of the option. This special property allows us to write the Black-Scholes formula for such an European call at time T as

$$C(S(T), \tau - T, \alpha S(T)) = S(T)\Lambda$$

$$\Lambda = N\left(\frac{-\ln\alpha + \left(r + \sigma^2/2\right)(\tau - T)}{\bar{\sigma}\sqrt{\tau - T}}\right)$$

$$-\alpha e^{-r(\tau - T)} N\left(\frac{-\ln\alpha + \left(r - \sigma^2/2\right)(\tau - T)}{\sigma\sqrt{\tau - T}}\right)$$

Forward Options cont.

Since Λ is a constant independent of the asset price, the pricing problem is obviously equivalent to the pricing of a futures contract at time *t*. The Future-Spot price parity implies

$$C_f(S,T,\tau,X) = \Lambda E^Q\left[S(T) \mid \mathcal{F}_0\right] = \exp\left(rT\right)\Lambda$$

European forward-start puts can be prices in the same way.

Ratchet Options

A ratchet option (a moving strike option or cliquet option) consists of a series of forward starting options where the strike price for the next exercise date is set equal to a positive constant times the asset price as of the previous exercise date.

For instance, a one-year ratchet call option with quarterly payments will normally have four payments (exercise dates) equal to the difference between the asset price and the strike price fixed at the previous exercise date. The strike price of the first option is usually set equal to the asset price of today.

A ratchet option can be priced as the sum of forward starting options:

$$P_{call} = \sum_{i=1}^{n} S e^{(b-r)t_i} \left[e^{-r(T_i - t_i)} N(d_1) - \alpha e^{-r(T_i - t_i)} N(d_2) \right]$$

where n is the number of settlements, t_i is the time to the forward start or strike fixing, and T_i is the time to maturity of the forward starting option.

A ratchet put is similar to a sum of forward starting puts.

Compound Options - Options on Options

The buyer of a compound option pays an initial premium for the right to pay a second set premium by a certain future date for the ownership of a call or a put with an agreed strike and expiry. These options are also known as **Instalment options.**

Compound options are options on options - i.e. the 'underlying' is another option. There are four types:

- Call on a call
- Call on a put
- Put on a call
- Put on a put

A model to price options on options was first given by Geske (1977). The model was enlarged and discussed by Hodges (1979), Selby (1987), and Rubenstein (1991).

Call on call

The payoff is given by: [BS(S, K_1, T_2) – K_2, 0]$^+$, where K_1 is the strike price of the underlying option, K_2 the strike price of the option on the option, and BS(S, K, T) is the Black-Scholes call option formula with strike K and time to maturity T.

$$P_{call} = Se^{(b-r)T_2} M\left(z_1, y_1, \rho\right) - K_1 e^{-rT_2} M\left(z_2, y_2, \rho\right) - K_2 e^{-rt} N\left(y_2\right)$$

$$y_1 = \frac{\ln\left(\frac{S}{I}\right) + \left(b + \frac{\sigma^2}{2}\right) t_1}{\sigma \cdot \sqrt{t_1}}, \qquad y_2 = y_1 - \sigma \cdot \sqrt{t_1}$$

$$\rho = \sqrt{\frac{t_1}{T_2}}$$

$$z_1 = \frac{\ln\left(\frac{S}{K_1}\right) + \left(b + \frac{\sigma^2}{2}\right) T_2}{\sigma \cdot \sqrt{T_2}}, \qquad z_2 = z_1 - \sigma \cdot \sqrt{T_2}$$

where T_2 is the time to maturity on the underlying option, and t_1 is the time to maturity on the option on the option. M is the cumulative bivariate normal distribution and I a critical value given by solving: BS(I, K_1, $T_2 - t_1$) = X_2. Similar formulas is given in Haug for put on call, call on put and put on put.

Multi-Asset Options

There are four major varieties of multi-asset option:

1. **Basket options**
2. **Outperformance options (exchange options** or **spread options)**
3. **"Better of" optios**: also known as a **rainbow options**
4. **Correlation options**

All the options described in this section may be priced using derivations of the analytical models, or by Monte Carlo simulation. A number of multi asset options is given in Haug.

Rainbow Options

Rainbow options refer to a family of options on the minimum or the maximum of two or more risky assets. Consider two assets:

$$\begin{cases} dS_1(t) = S_1(t)\mu_1(t)dt + S_1(t)\sigma_1(t)dW_1(t) \\ dS_2(t) = S_2(t)\mu_2(t)dt + S_2(t)\sigma_2(t)\rho(t)dW_1(t) + S_2(t)\sigma_2(t)\sqrt{1-\rho(t)^2}\,dW_2(t) \end{cases}$$

where σ_1, σ_2 and ρ are all deterministic processes, while μ_1 and μ_2 are predictable. The two standard Brownian motions W_1 and W_2 are independent, so the correlation coefficient between

$d \ln S_1(t)$ and $d \ln S_2(t)$ is $\rho(t)$. The two assets also generate continuous yields at deterministic rate processes θ_1 and θ_2, respectively. A call on the minimum of these two assets for a strike price $X > 0$ can be priced as:

Rainbow Options

$$C_{\min}(S_1,S_2,T,X)=e^{-rT}E^Q\left[\left(\min(S_1(T),S_2(T))-X\right)^+\right]$$
$$=e^{-rT}E^Q\left[S_1(T)\,|\,S_2(T)>X \quad \text{and} \quad S_2(T)>S_1(T)\right]$$
$$+e^{-rT}E^Q\left[S_2(T)\,|\,S_1(T)>X \quad \text{and} \quad S_1(T)>S_2(T)\right]$$
$$-e^{-rT}X\cdot E^Q\left[1\,|\,S_1(T)>X \quad \text{or} \quad S_2(T)>X\right]$$

The arbitrage free condition is equivalent to the existence of an equivalent measure *Q*, so that:

$$\begin{cases} S_1(T)=S_1(0)\exp\left\{rT-\int_0^T\theta_1(s)ds-\frac{1}{2}\int_0^T\sigma_1(s)^2ds+\int_0^T\sigma_1(s)dW_1(s)\right\} \\ S_2(T)=S_2(0)\exp\left\{rT-\int_0^T\theta_2(s)ds-\frac{1}{2}\int_0^T\sigma_2(s)^2ds\right\} \\ \qquad\times\exp\left\{\int_0^T\sigma_2(s)\rho(s)dW_1(s)+\int_0^T\sigma_2(s)\sqrt{1-\rho(s)^2}dW_2(s)\right\} \end{cases}$$

Rainbow Options cont.

The covariance matrix for [ln $S_1(T)$ ln $S_2(T)$]' is

$$\begin{bmatrix} \bar{\sigma}_1^2 & \bar{\rho}\bar{\sigma}_1\bar{\sigma}_2 \\ \bar{\rho}\bar{\sigma}_1\bar{\sigma}_2 & \bar{\sigma}_2^2 \end{bmatrix}$$

$$\begin{cases} \bar{\sigma}_1^2 = \dfrac{1}{T}\displaystyle\int_0^T \sigma_1(s)ds, \\ \bar{\sigma}_2^2 = \dfrac{1}{T}\displaystyle\int_0^T \sigma_2(s)ds \\ \bar{\rho} = \dfrac{1}{\bar{\sigma}_2 T}\displaystyle\int_0^T \sigma_2(s)\rho(s)ds \end{cases}$$

Noticing that the variance of ln $S_2(t)$ - ln $S_1(t)$ is

$$\bar{\sigma}^2 = \bar{\sigma}_1^2 + \bar{\sigma}_2^2 - 2\bar{\rho}\bar{\sigma}_1\bar{\sigma}_2,$$

Rainbow Options cont.

The correlation coefficient between ln $S_1(t)$ and ln $S_2(t)$ - ln $S_1(t)$ is, and that the correlation coefficient between ln $S_2(t)$ and ln $S_1(t)$ - ln $S_2(t)$ is , we evaluate the conditional expectation:

$$C_{\min}(S_1, S_2, T, X) = S_1 \exp\left(-\int_0^T \theta_1(s)ds\right) N_{biv}\left(h_1, h_3, \frac{\bar{\rho}\bar{\sigma}_2 - \bar{\sigma}_1}{\bar{\sigma}}\right)$$

$$+ S_2 \exp\left(-\int_0^T \theta_2(s)ds\right) N_{biv}\left(h_2, h_4, \frac{\bar{\rho}\bar{\sigma}_1 - \bar{\sigma}_2}{\bar{\sigma}}\right)$$

$$- e^{-rT} X \cdot N_{biv}\left(h_1 - \bar{\sigma}_1\sqrt{T}, h_3 - \bar{\sigma}_2\sqrt{T}, \bar{\rho}\right)$$

$$h_1 = \frac{\ln(S_1 / X) - \int_0^T \theta_1(s)ds + (r + \bar{\sigma}_1^2 / 2)T}{\bar{\sigma}_1\sqrt{T}}$$

Rainbow Options cont.

$$h_2 = \frac{\ln(S_2 / X) - \int_0^T \theta_2(s)ds + (r + \bar{\sigma}_2^2 / 2)T}{\bar{\sigma}_2\sqrt{T}}$$

$$h_3 = \frac{\ln(S_2 / S_1) + \int_0^T \left(\theta_1(s) - \theta_2(s)\right)ds + (r - \bar{\sigma}^2 / 2)T}{\bar{\sigma}\sqrt{T}}$$

$$h_4 = \frac{\ln(S_1 / S_2) + \int_0^T \left(\theta_2(s) - \theta_1(s)\right)ds + (r - \bar{\sigma}^2 / 2)T}{\bar{\sigma}\sqrt{T}}$$

Rainbow Options cont.

The key is to realize $C_{min}(S_1, S_2, T, 0) = \min(S_1, S_2)$ because the asset prices are always positive. Now, let us turn attention to options on the maximum. Its claim at T can be written as:

$$K(T) = \left(\max\left(S_1(T), S_2(T)\right) - X\right)^+$$
$$= \left[\left(S_1(T) - X\right) + \left(S_2(T) - X\right) - \left(\min\left(S_1(T), S_2(T)\right) - X\right)\right]^+$$
$$= \left(S_1(T) - X\right)^+ + \left(S_2(T) - X\right)^+ - \left(\min\left(S_1(T), S_2(T)\right) - X\right)^+$$

The last equality follows because the third term must cancel out one of the first two terms. Hence, a call on the maximum is equivalent to a long position in two regular calls and a short position in a call on minimum. Its price can be valued as

$$C_{\max}(S_1, S_2, T, X) = C(S_1, T, X) + C(S_2, T, X) - C_{\min}(S_1, S_2, T, X)$$

Rainbow Options cont.

Similarly, the claim of a put on maximum at time T is:

$$
\begin{aligned}
K(T) &= \left(X - \max\left(S_1(T), S_2(T)\right)\right)^+ \\
&= \left[\left(X - S_1(T)\right) + \left(X - S_2(T)\right) - \left(X - \min\left(S_1(T), S_2(T)\right)\right)\right]^+ \\
&= \left(X - S_1(T)\right)^+ + \left(X - S_2(T)\right)^+ - \left(X - \min\left(S_1(T), S_2(T)\right)\right)^+
\end{aligned}
$$

$$P_{\max}(S_1, S_2, T, X) = P(S_1, T, X) + P(S_2, T, X) - P_{\min}(S_1, S_2, T, X)$$

Pay-Later Options

A pay-later option is the right to buy (for a call) or sell (for a put) an asset at time $T > 0$ for a strike price $X > 0$ with the following features:

- The premium for this option is paid only on the exercise,
- The option must be exercised if the asset price is above (for a call) or below (for a put) X.

The price of a pay-later cal can be written as:

$$
\begin{aligned}
C_{pc}(S,T,X) &= e^{-rT}E^{Q}\left[\left(S(T)-X-X_c\right)|\,S(T)>X\right] \\
&= e^{-rT}E^{Q}\left[\left(S(T)-X\right)|\,S(T)>X\right]-X_c e^{-rT}E^{Q}\left[1|\,S(T)>X\right] \\
&= C(S,T,X)-X_c\cdot C_d(S,T,X)
\end{aligned}
$$

where X_c is the premium of the option. Thus, a pay-later call is a combination of a long position in a usual call and a short position in a digital call. Since it is costless to get such an option, its price must be zero.

Pay-Later Options cont.

This results in the exact value:

$$X_c = \frac{C(S,T,X)}{C_d(S,T,X)}$$

which remains constant throughout the life of the option. Accordingly, the price of a pay-later put is:

$$P_p(S,T,X) = P(S,T,X) - X_p \cdot P_d(S,T,X)$$

where the value of X_p can be solved by setting $P_p(S, T, X) = 0$:

$$X_p = \frac{P(S,T,X)}{P_d(S,T,X)}$$

Extendible Options

An extendible option grants the holder the right to extend the option to a later expiration time with a new strike price. Consider a call with a strike price $X_1 > 0$ and maturity $T_1 > 0$ when the holder can extend the option to time $T_2 > 0$ with a new strike price $X_2 > 0$ by paying a premium $A > 0$. The claim of this option at time T_1 is:

$$\begin{aligned} K(T_1) &= \max\left(0, C(S(T_1), T_2 - T_1, X_2) - A, S(T_1) - X_1\right) \\ &= \max\left(\left(C(S(T_1), T_2 - T_1, X_2) - A\right)^+, \left(S(T_1) - X_1\right)^+\right) \end{aligned}$$

This is a compound rainbow option. The first asset is a compound option while the second is a standard call. It is straightforward to price this call as

$$C_c(S, T_1, T_2, X_1, X_2, A) = C_{\max}\left(CC(S, T_1, T_2, X_2, A), C(S, T_1, X_1), T_1, 0\right)$$

$$P_c(S, T_1, T_2, X_1, X_2, A) = P_{\max}\left(PP(S, T_1, T_2, X_2, A), P(S, T_1, X_1), T_1, 0\right)$$

Quantos

Quantos are a family of contingent claims whose payoff are defined with respect to the value of some foreign asset in their own currency, but denominated in the domestic currency. The price of a foreign asset follows a diffusion process:

$$dS(t) = S(t)\mu_S(t)dt + S(t)\sigma_S(t)dW_1(t)$$

The exchange rate follows another diffusion process

$$dC(t) = C(t)\mu_C(t)dt + C(t)\sigma_C(t)dW_1(t) + C(t)\sqrt{1-\rho^2(t)}\sigma_C(t)dW_2(t)$$

where W_1 and W_2 are independent standard Brownian motions, σ_s, σ_c and ρ are deterministic processes, while μ_s and μ_c are both predictable. Thus, the correlation between $d\ln C(t)$ and $d\ln S(t)$ is $\rho(t)$. Moreover, the asset S is associated with a proportional dividend process θ. Let r and r_f be the domestic and the foreign deterministic interest rate processes, respectively.

Quantos cont.

In the domestic market, foreign assets are not directly tradable, but the foreign currency and the foreign asset value in the domestic market are. To eliminate the arbitrage opportunity:

$$\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \exp\left(-\int_0^T r(s)ds \right) \begin{bmatrix} \exp\left(\int_0^T r_f(s)ds \right) C(t) \\ \exp\left(\int_0^T \theta(s)ds \right) S(t)C(t) \end{bmatrix}$$

has to be martingale under the risk neutral measure Q.

Quantos cont.

Starting with

$$dY_1(t) = Y_1(t)\left(\mu_C(t) - r(t) + r_f(t)\right)dt + Y_1(t)\sigma_C(t)dW_1(t)$$
$$+ Y_1(t)\sqrt{1-\rho^2(t)}\sigma_C(t)dW_2(t)$$

$$dY_2(t) = Y_2(t)\left(\mu_C(t) + \mu_S(t) - r(t) + \theta(t)\right)dt + Y_2(t)\left(\sigma_S(t) + \rho(t)\sigma_C(t)\right)dW_1(t)$$
$$+ Y_2(t)\sqrt{1-\rho^2(t)}\sigma_C(t)dW_2(t)$$

we have

$$dC(t) = C(t)\left(r(t) - r_f(t)\right)dt + C(t)\rho(t)\sigma_C(t)d\tilde{W}_1(t)$$
$$+ C(t)\sqrt{1-\rho^2(t)}\sigma_C(t)d\tilde{W}_2(t)$$

$$dS(t) = S(t)\left(r(t) - \theta(t) - \rho(t)\sigma_S(t)\sigma_C(t)\right)dt + S(t)\sigma_S(t)d\tilde{W}_1(t)$$

under the martingale measure Q.

Quantos cont.

The foreign asset at time T under the measure Q becomes:

$$S(T) = S(0)\exp\left(-\int_0^T \theta(s)ds + \int_0^T \sigma_S(s)dW_1(s)\right)$$
$$\times \exp\left(\int_0^T r_f(s)ds - \int_0^T \rho(s)\sigma_S(s)\sigma_C(s)ds - \frac{1}{2}\int_0^T \sigma_S^2(s)ds\right)$$

Define

$$\bar{\sigma}_S^2 = \frac{1}{T}\int_0^T \sigma_S^2(s)ds$$

$$\bar{\sigma}_{CS}^2 = \frac{1}{T}\int_0^T \rho(s)\sigma_S(s)\sigma_C(s)ds$$

Without loss of generality, it is assumed that $C(0) = 1$. It follows:

$$S(0) = \frac{1}{B(T)} E^Q\left[C(T)S(T) \mid \mathcal{F}_0\right]$$

Quantos cont.

Since interest rates are deterministic, forward constraints are equivalent to futures contracts. Compared to the Future-Spot price parity for domestic securities, a quanto forward or futures price can be written as

$$f = F = \exp\left(-\int_0^T \theta(s)ds \right) E^Q\left[S(T) \mid \mathcal{F}_0 \right] = S\exp\left(\left(\bar{r}_f - \bar{\sigma}_{CS}^2 \right) T - \int_0^T \theta(s)ds \right)$$

Depending on the sign of the average covariance, the quanto futures price can be either greater or less than the standard futures price. It is correlated to the exchange process because the replicating portfolio involves the foreign currency and the foreign asset.

Next we turn into a digital call. Its price can be written as:

$$C_d(S,C,T,X) = e^{-\bar{r}T}\,\Pr\left[S(T) > X \right] = e^{-\bar{r}T} N(d)$$

$$d = \frac{\ln(S/X) - \int_0^T \theta(s)ds + \left(\bar{r}_f - \bar{\sigma}_{CS}^2 - \bar{\sigma}_S^2/2 \right) T}{\bar{\sigma}_S \sqrt{T}}$$

Quantos cont.

A quanto call is no more complicated. It can be priced as

$$C_q(S,C,T,X) = e^{-\bar{r}T} E^Q\left[\left(S(T)-X\right)^+\right]$$

$$= S\exp\left(-\int_0^T \theta(s)ds\right) e^{\left(\bar{r}_f - \bar{r} - \bar{\sigma}_{CS}^2\right)T} N\left(d - \bar{\sigma}_S\sqrt{T}\right) - Xe^{-\bar{r}T} N(d)$$

As we seen, the quanto option is designed to eliminate the FX risk to the buyer, so its payoff depends only on the performance of the foreign asset. However, the counterparty selling the quanto has to absorb the currency risk, so in pricing this option they must take into account not only the volatility of the exchange rate but also the correlation between it and the foreign asset price.

Other Exotic options

- Correlation options
- Exchange Options
- Currency-Linked Options
- Fixed Domestic Strike Options
- …..

Pricing using deflators

Consider a simple one-period model The price of the stock at the $t = 0$ is S_0, and S denotes the stochastic price at the end of the period.
In the Black-Scholes model, stock prices follow a lognormal distribution. Formally, the returns on the stock are generated by

$$\ln\left(\frac{S}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma \cdot \varepsilon\sqrt{t}$$

where ε is a standard normal random variable with mean zero and variance equal to one. The parameter μ equals the expected return, $E[S/S_0] = \mu$, and σ is the standard deviation of returns. The sharp quote or market price of risk:

$$\lambda = \frac{\mu - R_f}{\sigma}$$

Black-Scholes deflator

The deflator is a stochastic discount factor, i.e. a discount factor that varies with the random variables driving the stock returns. The deflator takes the form:

$$D = \exp\left(-R_f - \frac{1}{2}\lambda^2 - \lambda \cdot \varepsilon\right) \approx \left(\frac{1}{1+R_f}\right)\left(\frac{1}{1+\lambda \cdot \varepsilon + \lambda^2}\right)$$

The deflator is the product of the risk free discount factor and a stochastic term, which depends on the shocks to the stock price. The deflator can be used to calculate the value of derivatives of the stock price as

$$X_0 = E\left[DX\right] = E\left[Df(S)\right]$$

Black-Scholes deflator

The deflator is a stochastic discount factor, i.e. a discount factor that varies with the random variables driving the stock returns. We have already, almost derived the expression of the Black-Scholes deflator when we used the Girsanov transformation between the observable market probabilities, P and risk neutral probability measure, Q. The Girsanov kernel, $g(t)$ that took us from P to Q was the market price of volatility risk, i.e., $g(t) = \lambda$.

We then used that $dQ(t) = L(t)dP$, where

$$\begin{cases} dL(t) = g(t)L(t)dW(t) \\ L(0) = 1 \end{cases}$$

With the known solution to $L(t)$ (using Itô on $\ln(L)$):

$$L(t) = \exp\left\{ \int_0^t g(s)dX(s) - \frac{1}{2}\int_0^t g^2(s)ds \right\}$$

Black-Scholes deflator

we get

$$L(t) = \exp\left\{\int_0^t \frac{r-\mu}{\sigma} dW(s) - \frac{1}{2}\int_0^t \left(\frac{r-\mu}{\sigma}\right)^2 ds\right\} = \exp\left\{\frac{r-\mu}{\sigma} W(t) - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 t\right\}$$

$$= \exp\left\{-\lambda \cdot W(t) - \frac{1}{2}\lambda^2 t\right\}$$

which we can express like:

$$L(t) = \left\{-\lambda \cdot \varepsilon\sqrt{t} - \frac{1}{2}\lambda^2 t\right\}$$

This means that the discount factor e^{-rt} in Q can, in P be expressed as

$$\boxed{D = \exp\left(-\left(r + \frac{1}{2}\lambda^2\right)t - \lambda \cdot \varepsilon\sqrt{t}\right)}$$

This is our definition of the Black-Scholes deflator. As we can see, $1/D$ is a stochastic process with a normal distribution:

$$N\left[r + \lambda^2/2, \lambda^2(T-t)\right]$$

Black-Scholes deflator

$$E[D] = E\left[\exp\left(-rt - \frac{1}{2}\lambda^2 t - \lambda\varepsilon\sqrt{t}\right)\right] = \exp(-rt)\,E\left[\exp\left(-\frac{1}{2}\lambda^2 t - \lambda\varepsilon\sqrt{t}\right)\right]$$

$$= E\left[\exp\left(-\frac{1}{2}\lambda^2 t\right)\right]E\left[\exp(-\lambda\varepsilon\sqrt{t})\right]\exp(-rt) = \exp\left(-\frac{1}{2}\lambda^2 t\right)\exp\left(\frac{1}{2}\lambda^2 t\right)\exp(-rt)$$

$$= \exp(-rt)$$

$$E\left[e^{-\gamma X}\right] = \exp\left\{-\gamma m + \frac{1}{2}\gamma^2\sigma^2\right\} \qquad E\left[e^{-\gamma\varepsilon\sqrt{t}}\right] = \exp\left\{\frac{1}{2}\gamma^2 t\right\}$$

A risk-less cash flow, say F dollars, will therefore be valued by the standard present value formula

$$PV(\mathrm{F}) = E[DF] = E[D]F = \exp(-R_f)F$$

Black-Scholes deflator

$$S = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma \cdot \varepsilon\sqrt{t}\right)$$

$$E\left[DS\right] = E\left[\exp\left(-rt - \lambda\varepsilon\sqrt{t} - \frac{1}{2}\lambda^2 t\right)S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t + \sigma\varepsilon\sqrt{t}\right)\right]$$

$$= S_0 E\left[\exp\left(\mu t - rt - \frac{1}{2}\left\{\sigma^2 + \lambda^2\right\}t + \left\{\sigma - \lambda\right\}\varepsilon\sqrt{t}\right)\right]$$

$$= S_0 \exp\left(\mu t - rt - \frac{1}{2}\left\{\sigma^2 + \lambda^2\right\}t + \frac{1}{2}\left\{\sigma - \lambda\right\}^2 t\right)$$

$$= S_0 \exp\left(\mu t - rt - \sigma\lambda t\right) = S_0$$

$$\lambda = \frac{\mu - R_f}{\sigma}$$