Analytical Finance I - 2018

Jan Röman
http://janroman.dhis.org
jan@prosoftware.se

Phone: 0708-606 306 / 073-960 74 05

Literature: Books + Appendix and Glossary

http://janroman.dhis.org/AFI/

Recommended: *Hull; Option, Futures and Other Derivatives*


The Mathematics of Equity Derivatives,
Markets, Risk and Valuation
ANALYTICAL
FINANCE
VOLUME I
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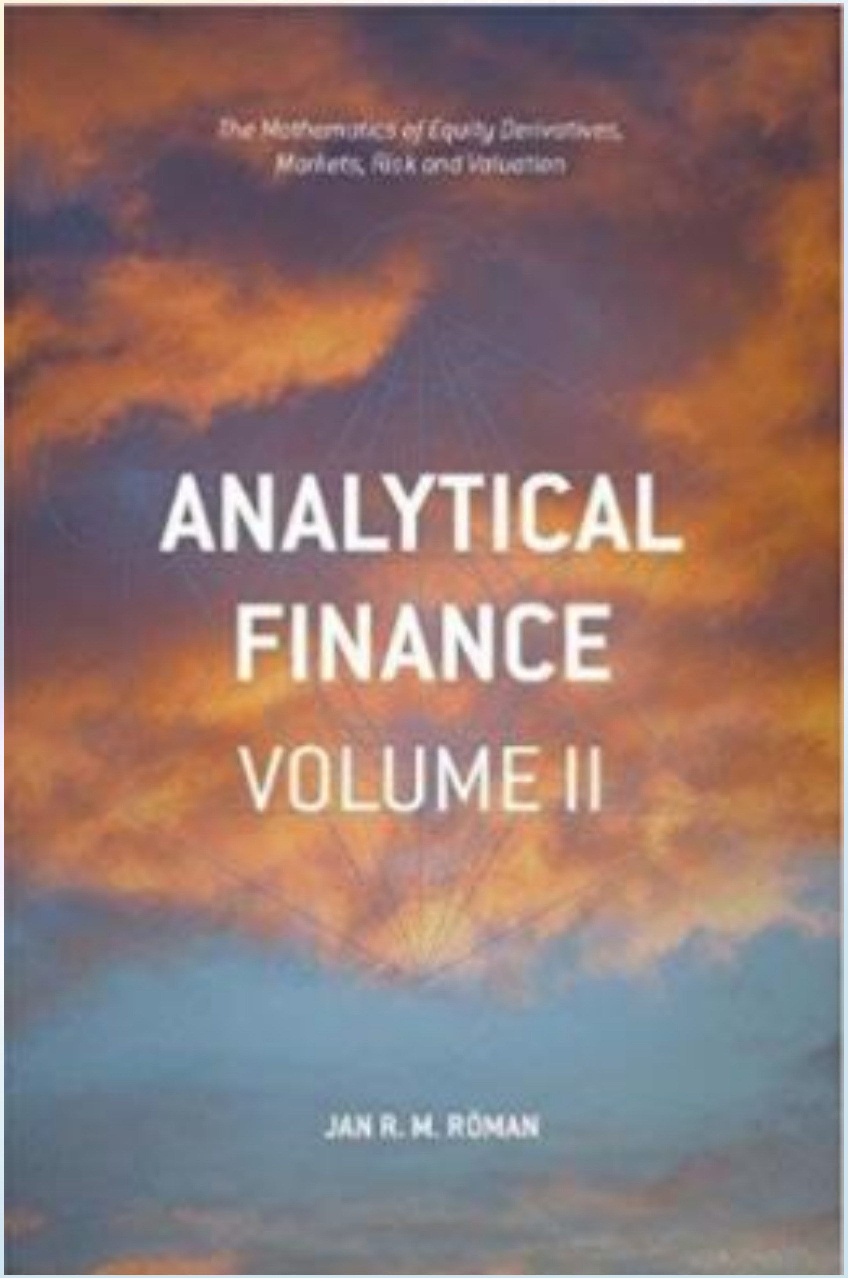
The Mathematics of Equity Derivatives,
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Introduction

All kind of financial instruments can be valued as the expected payoff in the future: $E[X(T)]$ where $X(T)$ is called a contingent claim.

This future value is discounted with some interest rate, r to give the present value of the claim.

With continuous compounding we can write the present value as of the contingent claim as:

$$X(t) = e^{-r(T-t)} E[X(T)].$$

Example: A call option on a stock with maturity T and strike price K, you will have the right, but not the obligation to buy the stock at time T to the price K.

Let $S(t)$ represents the stock price at time t, then

$$X(T) = \max\{S(T) - K, 0\}.$$

Introduction

The present value is

$$X(t) = e^{-r(T-t)} E[X(T)] = e^{-r(T-t)} E[\max\{S(T) - K, 0\}].$$

The max function indicate a price of zero if $K > S(T)$.

By solving this expectation value we will see that this can be given as the Black-Scholes formula. But generally we have a solution as:

$$X(t) = S(0) \cdot Q_1(S(T) > K) - e^{-r(T-t)} K \cdot Q_2(S(T) > K)$$

Where $Q_1(S(T) > K)$ and $Q_2(S(T) > K)$ is the probability for the underlying price to reach the strike price K in different "reference systems". This is simplified to the Black-Scholes formula as:

$$X(t) = S(t) \cdot N(d_1) - e^{-r(T-t)} K \cdot N(d_2).$$

Introduction

Here d_1 and d_2 are given (derived) variables.

$N(x)$ is the normal distribution function with mean 0 and variance 1

$N(d_2)$ represent the probability for the stock to reach the strike K.

The variables d_1 and d_2 will depends on

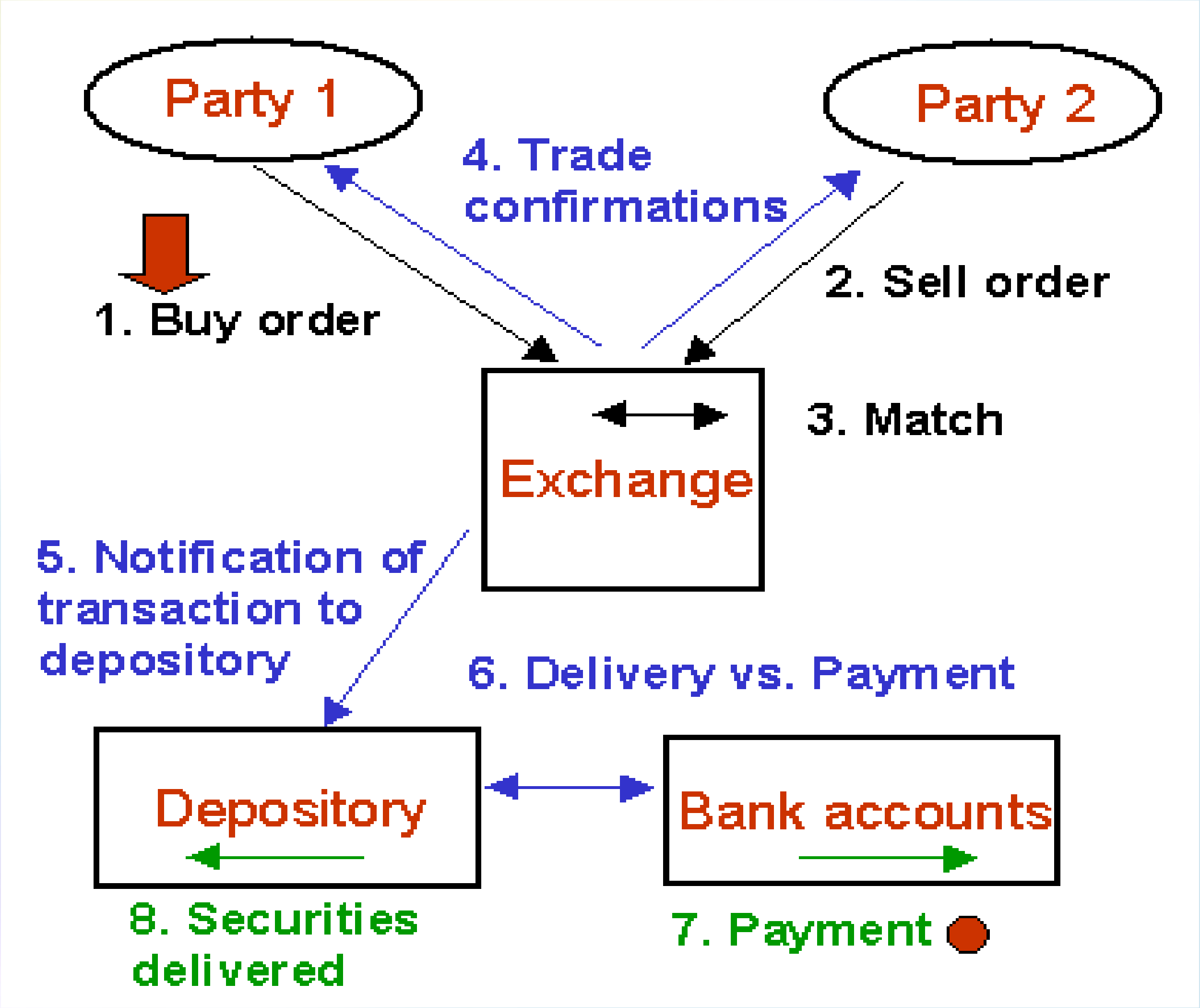
- the initial stock price,
- the strike price,
- the interest rate,
- The time to maturity and
- the volatility.

The volatility is a measure of how much the stock price may vary in a specific period in time. Normally we use 252 days, since this is an approximation on the number of trading days in a year.

Trading

- Exchange or OTC
- Clearing House
- Counterparties
- Risk
- Margin Requirement

Exchange trading



There are many kind of Risk

Market risk refers to the risk due to changes in interest rates, exchange rates and equity prices.

Liquidity risk refers to the risk that a bank cannot fulfil its payment commitments on any given date without significantly raising the cost.

Currency risk refers to the risk that the value of assets or liabilities may fluctuate due to changes in exchange rates.

Interest rate risk refers to the risk that the value of assets, liabilities and interest-related derivatives may be affected by changes in interest rate levels.

There are many kind of Risk

Credit risk is defined as the risk that a counterparty fails to meet his obligations and the risk that collateral will not cover the claim.

Credit Risk arises also when dealing in financial instruments, but this is often called **Counterparty risk.** The risk arises as an effect of the possible failure by the counterparty in a financial transaction to meet its obligations. This risk is often expressed as the current market value of the contract adjusted with an **add-on** for future potential movements in the underlying risk factors. This includes also **Settlement risk**.

Model risk refers to the possibility of loss due to errors in mathematical models, often models of derivatives. Since these models contain parameters, such as volatility, we can also speak of parameter risk, volatility risk, etc.

Operational risk refers to the risk of losses resulting from failed internal processes or routines, human error, incorrect systems or external events.

We have 2 cases:

Risks where the probabilities for events in the future are measurable and known, i.e., we have randomness, but with known probabilities. This can be further divided:

(a) A priori risk, such as the outcome of the roll of a dice, tossing coins etc.

(b) Estimable risk, where the probabilities can be estimated by statistical analysis of the past, for example, the probability of a one-day fall of ten percent in a stock index.

With uncertainty the probabilities of future events cannot be estimated or calculated.

How do we measure Risk?

In finance we tend to concentrate on risk with probabilities we estimate, we then have all the tools of statistics and probability for quantifying the risk.

In some financial models we address the uncertainty to the volatility. Here volatility is allowed to lie within a specified range, but the probability of the values is not given.

A starting point for a mathematical definition of risk is simply as standard deviation. This is sensible because of the results of the Central Limit Theorem where we add up a large number of investments so the resulting portfolio returns are normally distributed.

If we only have a small number of investments or, if the investments are correlated, or if they don't have finite variance, then standard deviation may not be relevant.

In the following, when we say risk, we mean the risk in volatility terms. I.e., the change in the underlying stock when calculate the value of a derivative.

Pricing via Arbitrage

Consider a simple financial market containing two instruments, *B* and *S*. We want to study a portfolio (*B*, *S*) today (time $t = 0$) and at a future time *t*. Here B has the following simple property:

$$B(0) = 1, \quad B(t) = 1 + r$$

This means; the value today is 1 (in some currency) and at *t*, the value is 1 + *r*, where *r* is **the risk-free interest rate**.

Now, on this market, two events may occur at time *t*: ω_1 or ω_2. We say that we have a **sample space** Ω with two possible **outcomes** $\Omega = \{\omega_1, \omega_2\}$. On event ω_1 the price of the security $S(t)$ will be $S_1(t)$ and on ω_2, $S_2(t)$. For simplicity, no other outcomes (events) exist.

We then have the following abstract representation of the situation in matrix representation:

$$\begin{pmatrix} B(0) \\ S(0) \end{pmatrix} = \begin{pmatrix} B(t) & B(t) \\ S_1(t) & S_2(t) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

or

$$\begin{pmatrix} B(0) \\ S(0) \end{pmatrix} = \begin{pmatrix} 1+r & 1+r \\ S_1(t) & S_2(t) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Pricing via Arbitrage

The first equation can be written as:

$$1 = B(0) = (1+r)\omega_1 + (1+r)\omega_2 = q_1 + q_2$$

where we have defined q_1 and q_2. Since the sum of q_1 and q_2 is equal to 1, we can interpret them as they are probabilities. We do not allow them to be less than zero. The second equation can then be written:

$$S(0) = S_1(t)\cdot\omega_1 + S_2(t)\cdot\omega_2 = \frac{1}{1+r}q_1\cdot S_1(t) + \frac{1}{1+r}q_2\cdot S_2(t)$$

$$= \frac{1}{1+r}\left[q_1\cdot S_1(t) + q_2\cdot S_2(t)\right]$$

We then say that under the **probability measure** $Q = (q_1, q_2)$, the value of S today (at time $t = 0$) is given by the **discounted expected payoff**. We write this as:

$$S(0) = \frac{1}{1+r}E^Q\left[S(t)\right] \quad \left(= e^{-rt}E^Q\left[S(t)\right]\right)$$

Remark! These probabilities have nothing to do with the real probability for the outcome in Ω. Therefore, we call these probabilities, **risk-adjusted probabilities**.

Pricing via Arbitrage

If we had used the true probabilities, *P* for the outcomes $\{\omega_1, \omega_2\}$, then

$$S(0) < \frac{1}{1+r} E^P\left[S(t)\right]$$

The reason is that those probabilities are not risk-free. If we are willing to buy a stock, which is more risky than a government bond (which pays a risk-free interest rate) we must be compensated for the higher risk. We say that we have a **risk premium** to go into the position of *S*:

$$S(0) = \frac{1}{1+r+\rho} E^P\left[S(t)\right]$$

This is the reason why we buy equities instead of risk-free bonds. We take the risk, since we hope we will get a better payoff. The expected payoff increases with the level of risk. Option have better payoff than stocks, since they are more risky.

Martingales

Expressions as the expectation value above will be frequently used in this course, especially when dealing with martingales. A martingale with respect to a given probability measure *Q*, is defined by:

$$E^Q\left[X(t+s) \mid I_t\right] = X(t)$$

for all $s > 0$. I_t is the information set that affects the value of the stochastic process *X*. This is also called a fair game. In words this expectation value is saying:

Standing at a time t, with a stochastic process X, under a given probability measure Q and a given information set I_t, (with information known up to time t), the calculated expected future value of $X(t + s)$ (where $s > 0$) is equal to $X(t)$.

If we have:

$$E^P\left[X(t+s) \mid I_t\right] \leq X(t)$$

we say that *X* is a **super-martingale** and if

$$E^P\left[X(t+s) \mid I_t\right] \geq X(t)$$

X is said to be a **sub-martingale**.

The Binomial Model

Consider a financial market during one period in time, from t = 0 to t = 1 with two possible investments, B and S. Here B represents a deterministic **money-market account** (or in some literature, a bond) with the price process:

$$\begin{cases} B(0) = 1 \\ B(1) = 1 + r \end{cases}$$

where r represent the **interest rate**. S is considered to be a **stock** with a stochastic price process given by:

$$\begin{cases} S(0) = s \\ S(1) = \begin{cases} u \cdot s & \text{with probability } q_u \\ d \cdot s & \text{with probability } q_d \end{cases} \end{cases}$$

At time t = 1 the stock can reach two possible value $u \cdot s$ where $u > 1$ or $d \cdot s$ where $d < 1$. I.e., the stock price can either increase or decrease with probability q_u and q_d . Here $q_u + q_d = 1$.

Furthermore, we suppose that we can buy (**go long**) or sell (**go short**) the stock and we can invest (put money, i.e., go long) or lend (borrow money, i.e., go short) in the money-market account. The interest rate for saving and lending money from the money-market account is the same, r.

Now, we write $S(t) = Z \cdot s$ where Z is a **stochastic variable**.

The Binomial Model

Now, consider a **portfolio**, h on the (B, S)-market, as a vector $h = (x, y)$ where x is the number money and y the number of stocks. x and y can take any number, including negative and fractions where negative values represent short positions. We also suppose that the market is 100% liquid, i.e., we can trade whenever we want.

Definition: The **value process** of the portfolio h is defined as:

$$V(t,h) = x \cdot B(t) + y \cdot S(t); \quad t = 0,1$$

i.e.

$$\begin{cases} V(0,h) = x + y \cdot s \\ V(1,h) = x \cdot (1+r) + y \cdot s \cdot Z \end{cases}$$

Definition: An **Arbitrage portfolio** of h is defined as:

$$\begin{cases} V(0,h) = 0 \\ V(1,h) > 0 \quad \text{with probability 1} \end{cases}$$

This means that we can borrow money at time $t = 0$ and buy the stock, or sell the stock and put the money at the money-market account. The total value of our portfolio h is then at time $t = 0$ equal zero. If for sure (with probability 1) our portfolio at time $t = 0$ have a value greater than zero we have made arbitrage.

The portfolio is **free of arbitrage** if and only if $d \leq 1 + r \leq u$. Why?

The Binomial Model

If $d \leq u \leq 1 + r$ we can go short in the stock and invest in the risk free interest rate.

If $1 + r \leq d \leq u$ we can go short in the risk free interest rate and invest in the stock.

From now on we denote the objective (true) probabilities as $P = (p_u, p_d)$ and the risk-free probabilities as $Q = (q_u, q_d)$. If the portfolio is risk-free we must have probabilities $Q = (q_u, q_d)$ such as:

$$1 + r = u \cdot q_u + d \cdot q_d; \quad q_u + q_d = 1$$

We then have

$$\frac{1}{1+r} E^Q\left[S(1)\right] = \frac{1}{1+r}\left(u \cdot S(0) \cdot q_u + d \cdot S(0) \cdot q_d\right) = \frac{1}{1+r} \cdot S(0) \cdot \left(1+r\right) = S(0)$$

or

$$S(0) = \frac{1}{1+r} E^Q\left[S(1)\right]$$

This is called the **risk-neutral valuation formula** and will be widely used. Q is called the **risk-neutral probability measure** or the **martingale measure**. If we use continuous compounded interest rate for a security with maturity T we write:

$$S(t) = e^{-r(T-t)} \cdot E^Q\left[S(T)\right]$$